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# Fuzzy Linear Programming Method for Deriving Priorities in the Fuzzy Analytic Hierarchy Process

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**Abstract** There are various methods for obtaining the preference vector of pair-wise comparison matrix factors. These methods can be employed when the elements of pair-wise comparison matrix are crisp while they are inefficient for fuzzy elements of pair-wise comparison matrix. In this paper, a method is proposed by which the preference vector of pair-wise comparison matrix elements can be obtained even if these elements are fuzzy. First we describe the method for the case of crisp pair-wise comparison matrix and then extend it to the case in which the pair-wise comparison matrix elements are fuzzy, finally, conclusion will be provided.

**Keywords** AHP · Fuzzy AHP · Linear Programming

**Mathematics Subject Classification (2010)** 90C05 · 90C70

## 1 Introduction

Employing scientific tools doe decision making is one of the necessities of today's life. Proper method for decision making is inevitable especially when the decision are highly important and sensitive in a way that improper decisions will result in non-compensable consequences.

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Numerous methods have been developed for Multiple-criteria decision making (MCDM) but AHP evaluation method, proposed by Mr Saaty in 1980 [4], has found considerable attention as it can be employed easily in decision making processes.

This method is based on pair-wise comparison of semi-level elements which facilitates consideration of different criteria especially those contradicting ones and different quantitative and qualitative criteria.

The main step in this method of evaluation is to determine the priority of semi-level elements (the elements on the same level of hierarchy) based on the criteria of the upper level. Mr Saaty proposed application of crisp numbers (1-9) for determination of priority of elements; he proposed 1, 3, 5, 7 and 9 for the same, low, strong, very strong and complete priority. He also proposed the numbers between the mentioned numbers for the intermediate states. Due to ambiguity in comparison, fuzzy numbers can be also employed.

The estimation of priorities from pairwise comparison matrices is the major constituent of the AHP. The priority vector can be derived from the comparison matrices using different techniques. The traditional method, proposed by Saaty is the Eigenvector Method (EVM) [4]. Saaty proves that the principal eigenvector of the comparison matrix can be used as a priority vector for consistent and inconsistent preferences. Most other methods for deriving priorities in the AHP are based on some optimisation approach, such as Direct Least Squares Method (DLSM), minimising the Euclidean distance from the given comparison matrix under additive normalisation constraints and the Weighted Least Squares Method (WLSM), using a modified Euclidean norm as an objective function [1]. The Logarithmic Least Squares Method (LLSM) of Crawford and Williams [2] makes use of the multiplicative properties of the pairwise comparison matrices and applies an optimisation procedure to minimise a logarithmic objective function, subject to multiplicative constraints. This method gives an explicit solution, which is rather simple and convenient from computational point of view. In [3] the author uses Fuzzy programming method (FPM) for priorities derivation from pairwise comparison matrices. The FPM is based on a geometrical representation of the prioritisation process as an intersection of fuzzy hyperlines and determines the values of the priorities, corresponding to the point with the highest measure of intersection. The reminder of this paper is organized as follows. In section 2 of this paper, a method will be introduced for obtaining the preference vector of pairwise comparison matrix with crisp numbers. This method will be generalized for fuzzy numbers in section 3 and conclusion will be presented in section 4.

## 2 Pair-wise comparison matrix with crisp numbers

Assume a square comparison matrix whose order depends on the number of compared objects. For example if we compare $n$ factors of $A_1, A_2, \dots, A_n$ relative to $X$ criterion, then the comparison matrix will be:

Table 1: comparison matrix

| $X$ | $A_1$ | $\cdots$ | $A_i$ | $\cdots$ | $A_n$ |
|---|---|---|---|---|---|
| $A_1$ | 1 | $\cdots$ | $a_{1i}$ | $\cdots$ | $a_{1n}$ |
| $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\cdots$ | $\vdots$ |
| $A_i$ | $a_{i1}$ | | 1 | | |
| $\vdots$ | $\vdots$ | | $\vdots$ | $\ddots$ | |
| $A_n$ | $a_{n1}$ | $\cdots$ | $a_{ni}$ | $\cdots$ | 1 |

And $a_{ij} = \frac{1}{a_{ji}}$, In which $a_{ij}$ shows the priority of element $A_i$ to $A_j$ in relation to $X$ criterion. Therefore, if element $A_i$ has the value of $w_i$ for $X$ criterion, then:

$$a_{ij} = \frac{w_i}{w_j} \tag{1}$$

**Theorem 1** *comparison matrix 1 is compatible if and only if:*

$$a_{1i} \times a_{ij} = a_{1j} \qquad i = 1, 2, \ldots, n, \ j = i + 1, \ldots, n \tag{2}$$

In the other words, pair-wise comparison matrix is compatible if and only if at least one of the above equations does not hold for the matrix elements. We are about to propose a method that in addition to computing the extent of matrix incompatibility, obtains the extent of incompatibility for any of equation 2 relations. For example consider below pair-wise comparison matrix which is incompatible:

$$\begin{bmatrix} 1 & 2 & 5 \\ \frac{1}{2} & 1 & 4 \\ \frac{1}{5} & \frac{1}{4} & 1 \end{bmatrix}$$

If we use the eigenvector method of Mr Saaty for weight calculation, the weights of the comparison matrix elements will be:

$$w_1 = 0.57 \qquad w_2 = 0.333 \qquad w_3 = 0.097$$

These weights will give us this compatible matrix ($a_{ij} = \frac{w_i}{w_j}$)

$$\begin{bmatrix} 1 & 1.7101 & 5.849 \\ 0.585 & 1 & 3.42 \\ 0.171 & 0.292 & 1 \end{bmatrix}$$

As it can be seen, the obtained compatible matrix is almost equal with the initial matrix as its elements are close to the elements of the initial matrix. In the method of prioritization by fuzzy programming, first the elements of a pair-wise comparison matrix which are crisp , will be transformed to fuzzy numbers; then by equation 2, the elements of the closest compatible matrix to the current matrix will be calculated. For easier application of equation 2, we employed $m_{ij} = \ln(a_{ij})$ to linearize the equations.

$$m_{1i} + m_{ij} = m_{1j} \qquad i = 1, 2, \ldots, n-1,\ j = i+1, \ldots, n \tag{3}$$

Therefore, a pair-wise comparison matrix is compatible if and only if equations 3 are correct. Therefore, instead of making $a_{ij}$ fuzzy, we make $m_{ij}$ fuzzy. Because if $\frac{w_i}{w_j}$ is in the range of $a_{ij}$, $\ln(\frac{w_i}{w_j})$ will be also in the range of $m_{ij}$. So, we considered $m_{ij}$ as triangular fuzzy numbers with left and right tolerances of $d_{ij}^-$ and $d_{ij}^+$ which are positive arbitrary numbers. Therefore the fuzzy number of $\tilde{m}_{ij}$ will be:

$$\tilde{m}_{ij} = \{(x, \mu_{\tilde{m}_{ij}}(x)) \mid x \in R\},$$

where

$$\mu_{\tilde{m}_{ij}}(x) = \begin{cases} 0 & x \leq m_{ij} - d_{ij}^- \\ \frac{1}{d_{ij}^-}(x - m_{ij}) + 1 & m_{ij} - d_{ij}^- \leq x \leq m_{ij} \\ 1 - \frac{1}{d_{ij}^+}(x - m_{ij}) & m_{ij} \leq x \leq m_{ij} + d_{ij}^+ \\ 0 & x \geq m_{ij} + d_{ij}^+ \end{cases} \tag{4}$$

So we have:

$$\tilde{m}_{1i} + \tilde{m}_{ij} \approx \tilde{m}_{1j} \qquad i = 1, 2, \ldots, n-1,\ j = i+1, \ldots, n \tag{5}$$

Above system has $m = \frac{(n-1)(n-2)}{2}$ equations. Assume that $k^{\text{th}}$ equation has the following from ($1 \leq k \leq m$):

$$\tilde{m}_{1i} + \tilde{m}_{ij} \approx \tilde{m}_{1j}$$

If $x_{1i}$ with member degree of $\mu_{\tilde{m}_{1i}}(x_{1i})$ belongs to fuzzy number of $\tilde{m}_{1i}$ and if $x_{ij}$ with member degree of $\mu_{\tilde{m}_{ij}}(x_{ij})$ and $x_{1j}$ with member degree of $\mu_{\tilde{m}_{1j}}(x_{1j})$ belong to fuzzy number of $\tilde{m}_{ij}$ and $\tilde{m}_{1j}$, respectively. In a way that $x_{1i} + x_{ij} = x_{1j}$, the extent of compatibility of $kth$ equation, $\mu_k$ will be defined as:

$$\mu_k = \min\{\mu_{\tilde{m}_{1i}}(x_{1i}), \mu_{\tilde{m}_{ij}}(x_{ij}), \mu_{\tilde{m}_{1j}}(x_{1j})\}$$

The total compatibility of pair-wise comparison matrix can be considered as the average of equations compatibilities, hence:

$$\mu_T = \frac{1}{m} \sum_{k=1}^{m} \mu_k$$

In which $m = \frac{(n-1)(n-2)}{2}$ and $n$ is the number of pair-wise comparison matrix factors.

The compatibility rate of pair-wise comparison matrix, $\mu_T$, has to be maximized. Therefore, we consider the following programming problem:

$$\begin{aligned}
&\max \textstyle\sum_{k=1}^{m} \mu_k \\
&s.t. \\
&\quad \mu_k \le \mu_{\tilde{m}_{1i}}(x_{1i}) \\
&\quad \mu_k \le \mu_{\tilde{m}_{ij}}(x_{ij}) \\
&\quad \mu_k \le \mu_{\tilde{m}_{1j}}(x_{1j}) \\
&\quad x_{1i} + x_{ij} = x_{1j} \;\; i = 1, 2, \ldots, n-1,\; j = i+1, \ldots, n.
\end{aligned}$$

Assume that is a triangular fuzzy number in which $d_{ij}^{+} = d_{ij}^{+} = 1$; this means that its membership function will have the following form, based on (4):

$$\tilde{m}_{ij} = \{(x, \mu_{\tilde{m}_{ij}}(x)) \mid x \in R\},$$

where

$$\mu_{\tilde{m}_{ij}}(x) = \begin{cases} 0 & x \le m_{ij} - 1 \\ 1 - |x - m_{ij}| & m_{ij} - 1 \le x \le m_{ij} + 1 \\ 0 & x \ge m_{ij} + d_{ij}^{+} \end{cases} \tag{6}$$

Hence, we have:

$$\begin{aligned}
&\max \textstyle\sum_{k=1}^{m} \mu_k \\
&s.t. \\
&\quad \mu_k + x_{1i} \le 1 + m_{1i} \\
&\quad \mu_k - x_{1i} \le 1 - m_{1i} \\
&\quad \mu_k + x_{ij} \le 1 + m_{ij} \\
&\quad \mu_k - x_{ij} \le 1 - m_{ij} \\
&\quad \mu_k + x_{1j} \le 1 + m_{1j} \\
&\quad \mu_k - x_{1j} \le 1 - m_{1j} \\
&\quad x_{1i} + x_{ij} = x_{1j} \qquad i = 1, 2, \ldots, n-1,\; j = i+1, \ldots, n.
\end{aligned}$$

By solving the programming problem, $\mu_k$ and $x_{ij}$ will be obtained, if we put , $\frac{w_i}{w_j} = e^{x_{ij}}$ approximate values of $a_{ij}$ will be determined.

*Example 1* Assume that pair-wise comparison matrix has the following from:

$$\begin{bmatrix} 1 & 2 & a_{13} \\ \frac{1}{2} & 1 & 3 \\ \frac{1}{a_{13}} & \frac{1}{3} & 1 \end{bmatrix}$$

In which $0.01 < a_{13} < 100$, therefore, according to the mentioned method we have: $\tilde{m}_{12} + \tilde{m}_{23} = \tilde{m}_{13}$ where

$$\tilde{m}_{12} = \{(x, \mu_{\tilde{m}_{12}}(x)) \mid x \in R\},$$

with

$$\mu_{\tilde{m}_{12}}(x) = \begin{cases} 1 - |x - 0.693| & 0.307 \le x \le 1.693 \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

Table 2: The results of estimation priorities from pairwise comparison matrix ($a_{13} = 5$)

| Methods | $w_1$ | $w_2$ | $w_3$ | $\frac{w_1}{w_2}$ | $\frac{w_1}{w_3}$ | $\frac{w_2}{w_3}$ | E |
|---|---|---|---|---|---|---|---|
| FPM | 0.583 | 0.306 | 0.111 | 1.909 | 5.250 | 2.750 | 0.365 |
| EVM | 0.582 | 0.309 | 0.109 | 1.882 | 5.313 | 2.823 | 0.379 |
| LLSM | 0.582 | 0.309 | 0.109 | 1.882 | 5.313 | 2.823 | 0.379 |
| WLSM | 0.585 | 0.302 | 0.113 | 1.937 | 5.189 | 2.679 | 0.377 |
| Fuzzy | 0.581 | 0.309 | 0.109 | 1.881 | 5.312 | 2.823 | 0.379 |

$$\tilde{m}_{23} = \{(x, \mu_{\tilde{m}_{23}}(x)) \mid x \in R\},$$

with

$$\mu_{\tilde{m}_{23}}(x) = \begin{cases} 1 - |x - 1.099| & -0.099 \leq x \leq 2.099 \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

and

$$\tilde{m}_{13} = \{(x, \mu_{\tilde{m}_{13}}(x)) \mid x \in R\},$$

with

$$\mu_{\tilde{m}_{13}}(x) = \begin{cases} 1 - |x - \ln(a_{13})| & \ln(a_{13}) \leq x \leq 1 + \ln(a_{13}) \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

Therefore the linear programming problem (8) in this example has the below form:

$$\begin{array}{l} \max \mu_1 \\ \quad s.t. \\ \qquad \mu_1 + x_{12} \leq 1.693 \\ \qquad \mu_1 - x_{12} \leq 0.307 \\ \qquad \mu_1 + x_{23} \leq 2.099 \\ \qquad \mu_1 - x_{23} \leq -0.099 \\ \qquad \mu_1 + x_{13} \leq 1 + \ln(a_{13}) \\ \qquad \mu_1 - x_{13} \leq \ln(a_{13}) - 1 \\ \qquad x_{12} + x_{23} = x_{12} \end{array}$$

First, we performed above method for $a_{13} = 5$ and compare the obtained values with those obtained from other methods. Then we investigate the problem with variable values of that with eigenvector method of Mr. Saaty.

If $a_{13} = 5$, then the solution for the linear programming problem is:

$$\mu_1 = 1.88137, \quad x_{12} = 0.632, \quad x_{13} = 1.67, \quad x_{23} = 1.038$$

and

$$\frac{w_1}{w_2} = 1.88137, \quad \frac{w_1}{w_3} = 5.3122, \quad \frac{w_2}{w_3} = 2.8236$$

As it can be seen, the results of this method are similar with the results of Saaty eigenvector method (see Table 2).

Now if we change $0.01 < a_{13} < 100$, we have estimation of priorities from pairwise comparison matrix in Table 3.

Table 3: The results of estimation priorities from pairwise comparison matrix ($0.01 \leq a_{13} \leq 100$)

| | Fuzzy | | | EVM | | | | |
|---|---|---|---|---|---|---|---|---|
| $a_{13}$ | $\frac{w_1}{w_2}$ | $\frac{w_1}{w_3}$ | $\frac{w_2}{w_3}$ | $\frac{w_1}{w_2}$ | $\frac{w_1}{w_3}$ | $\frac{w_2}{w_3}$ | $\mu_T$ | C.R. |
| 0.01 | 0.237 | 0.084 | 0.3356 | 0.237 | 0.084 | 0.356 | -1.323 | 5.649 |
| 0.1 | 0.511 | 0.391 | 0.766 | 0.511 | 0.392 | 0.766 | -0.365 | 1.871 |
| 1 | 1.100 | 1.817 | 1.651 | 1.100 | 1.817 | 1.651 | 0.403 | 0.317 |
| 2 | 1.387 | 2.883 | 2.081 | 1.387 | 2.885 | 2.081 | 0.634 | 0.117 |
| 3 | 1.587 | 3.781 | 2.382 | 1.588 | 3.782 | 2.382 | 0.769 | 0.046 |
| 4 | 1.747 | 4.577 | 2.622 | 1.747 | 4.580 | 2.621 | 0.865 | 0.016 |
| 5 | 1.881 | 5.312 | 2.823 | 1.882 | 5.315 | 2.824 | 0.939 | 0.003 |
| 6 | 2 | 6 | 3 | 2 | 6 | 3 | 1 | 0 |
| 7 | 2.104 | 6.652 | 3.158 | 2.106 | 6.650 | 3.158 | 0.949 | 0.002 |
| 8 | 2.201 | 7.264 | 3.304 | 2.201 | 7.274 | 3.304 | 0.904 | 0.0159 |
| 9 | 2.288 | 7.862 | 3.435 | 2.289 | 7.867 | 3.436 | 0.865 | 0.0157 |
| 10 | 2.370 | 8.44 | 3.557 | 2.371 | 8.435 | 3.557 | 0.830 | 0.025 |
| 20 | 2.986 | 13.397 | 4.482 | 2.988 | 13.398 | 4.484 | 0.599 | 0.140 |
| 100 | 5.108 | 39.134 | 7.668 | 5.107 | 39.087 | 7.6534 | 0.0623 | 0.815 |

We used two method of fuzzy programming and eigenvalue method of EVM. We can see that the difference between the results is negligible and based on the above table; the compatibility of pair-wise comparison matrix varies with variation of $a_{13}$.

## 3 Pair-wise comparison matrix with fuzzy numbers

Fuzzy programming method addresses the transformation of pair-wise comparison matrix elements to fuzzy numbers to calculate the closest compatible matrix to the current one and as we observed, this method has the accuracy the same as Mr. Saaty method and can be used for the state in which the pair-wise comparison matrix elements are fuzzy to obtain the closest compatible matrix. It must be noted that the fuzzy numbers of the matrix may have different types which would result in nonlinear constraints for optimization problem that can be solved by nonlinear optimization methods.

But if the fuzzy numbers are triangular or trapezoidal, the programming problem could be transformed to a linear programming problem. Example: assume that the pair-wise comparison matrix has the following form:

$$\begin{bmatrix} \tilde{1} & \tilde{2} & \tilde{2} \\ \frac{\tilde{1}}{2} & \tilde{1} & \tilde{3} \\ \frac{\tilde{1}}{2} & \frac{\tilde{1}}{3} & \tilde{1} \end{bmatrix}$$

In which $a_{13}$ and $a_{12}$ are triangular fuzzy numbers with following definition:

$$a_{12} = a_{13} = \{(x, \mu_{\tilde{2}}(x)) \mid x \in R\}$$

$$\mu_{\tilde{2}}(x) = \begin{cases} 0 & x \leq 1 \\ 1 - |x-2| & 1 \leq x \leq 3 \\ 0 & x \geq 3 \end{cases} \tag{10}$$

$$a_{23} = \{(x, \mu_{\tilde{3}}(x)) \mid x \in R\}$$

$$\mu_{\tilde{3}}(x) = \begin{cases} 0 & x \leq 2 \\ 2x-4 & 2 \leq x \leq 2.5 \\ 1 & 2.57 \leq x \leq 3.5 \\ 8-2x & 2.5 \leq x \leq 4 \\ 0 & x \geq 4 \end{cases} \tag{11}$$

As mentioned before, for having a compatible matrix it must:

$$\tilde{m}_{12} + \tilde{m}_{23} \approx \tilde{m}_{13}.$$

In which $\tilde{m}_{ij} = \ln(\tilde{m}_{ij})$ by this relation, the equations can be linearized and finally we will have the following linear programming problem:

$$\begin{array}{l} \max \mu_1 \\ \quad s.t. \\ \qquad 1.099\mu_1 + x_{12} \leq 2.198 \\ \qquad 1.099\mu_1 - x_{12} \leq 0 \\ \qquad 1.099\mu_1 + x_{13} \leq 2.198 \\ \qquad 1.099\mu_1 - x_{13} \leq 0 \\ \qquad \mu_1 - 4.484x_{23} \leq -3.107 \\ \qquad \mu_1 \leq 1 \\ \qquad \mu_1 + 3.484x_{23} \leq 4.829 \\ \qquad x_{12} + x_{23} = x_{12} \end{array}$$

By solving this programming problem we have:

$$\mu_1 = 0.622, \quad x_{12} = 0.683, \quad x_{13} = 1.515, \quad x_{23} = 0.831,$$

$$\frac{w_1}{w_2} = 1.980, \quad \frac{w_1}{w_3} = 4.549, \quad \frac{w_2}{w_3} = 2.295,$$

and

$$w_1 = 0.575, \quad w_2 = 0.296, \quad w_3 = 0.575.$$

## 4 Conclusion

One of the other advantages of the proposed fuzzy method in the previous section for determination of the weights of the pair-wise comparison matrix elements is that the extent of equations (1) compatibility, which is important in matrix compatibility, will be calculated and it determines a series of equations (1) which enhance the incompatibility of matrix. Increase of matrix dimension will increase its incompatibility as the decision maker should do more comparisons which increase the possibility of error in matrix elements estimation. Therefore, a series of matrix elements which are playing crucial role in reduction of matrix compatibility can be determined. This will provide the decision maker with better insight on pair-wise comparison matrix elements estimation.

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