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Semi-amenability and Connes Semi-amenability of Banach Algebras

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Abstract Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ a Banach $\mathcal{A}$-bimodule, the derivation $\mathcal{D} : \mathcal{A} \to \mathcal{X}$ is *semi-inner* if there are $\xi, \mu \in \mathcal{X}$ such that $\mathcal{D}(a) = a.\xi - \mu.a$, $(a \in \mathcal{A})$. $\mathcal{A}$ is called semi-amenable if every derivation $\mathcal{D} : \mathcal{A} \to \mathcal{X}^*$ is semi-inner. The dual Banach algebra $\mathcal{A}$ is *Connes semi-amenable* (resp. *approximately semi-amenable*) if, every $\mathcal{D} \in \mathcal{Z}^1_{w^*}(\mathcal{A}, \mathcal{X})$, for each normal, dual Banach $\mathcal{A}$-bimodule $\mathcal{X}$, is semi -inner (resp. approximately semi-inner). We will investigate on some properties of semi-amenability and Connes semi-amenability of Banach algebras which former have been studied for amenability case.

Keywords Amenability · Semi-amenablity · Connes-amenability · Connes semi-amenability

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1 Introduction

The systematic study of the notion of amenability has its origin in the beginning of the modern measure theory in the earlier part of the twentieth century. It is worthwhile to say that the theory of amenable Banach algebras begins from B. E. Johnson's memoir [7], and the choice of terminology comes from Theorem 2.5 in [7]. The purpose of this note is to give an overview of what has been done so far on semi-amenability and various notions of semi-amenability and raise some problems in semi-amenability and Connes semi-amenability of Banach algebras.
First, we recall some standard notions; for further details, see [3, 4, 5, 11].
Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ a Banach $\mathcal{A}$-bimodule. A bounded linear mapping $\mathcal{D} : \mathcal{A} \to \mathcal{X}$ is a derivation if $\mathcal{D}(ab) = \mathcal{D}(a)b + a\mathcal{D}(b)$, for all $a, b \in \mathcal{A}$. For each $x \in \mathcal{X}$, the mapping $\delta_x(a) = ax - xa$, $(a \in \mathcal{A})$ is a continuous derivation, called an inner derivation. A Banach algebra $\mathcal{A}$ is called amenable (resp. contractible) if for each Banach $\mathcal{A}$-bimodule $\mathcal{X}$ every continuous derivation $\mathcal{D}$ from $\mathcal{A}$ into $\mathcal{X}^*$ (resp. into $\mathcal{X}$) is inner i.e. $\mathcal{H}^1(\mathcal{A}, \mathcal{X}^*) = \{0\}$ (*resp.* $\mathcal{H}^1(\mathcal{A}, \mathcal{X}) = \{0\}$), where $\mathcal{H}^1(\mathcal{A}, \mathcal{X}^*)$ is the first cohomology group of $\mathcal{A}$ with coefficients in $\mathcal{X}^*$ [7].

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. A derivation $\mathcal{D} : \mathcal{A} \to \mathcal{X}$ is *approximately inner* if there exits a net $(\xi_\alpha)_\alpha$ in $\mathcal{X}$, such that $\mathcal{D}(a) = \lim_\alpha(a.\xi_\alpha - \xi_\alpha.a)$, $(a \in \mathcal{A})$ where the limit being in norm topology of $\mathcal{X}$ for more information see [5]. A Banach algebra $\mathcal{A}$ is approximately amenable (resp. approximately contractible) if for any Banach $\mathcal{A}$-bimodule $\mathcal{X}$, every derivation from $\mathcal{A}$ into $\mathcal{X}^*$ (resp. into $\mathcal{X}$) is approximately inner, for more details see [5].
For Banach algebra $\mathcal{A}$ and Banach $\mathcal{A}$-bimodule $\mathcal{X}$ the derivation $\mathcal{D} : \mathcal{A} \to \mathcal{X}$ is *semi-inner* if there are $\xi, \mu \in \mathcal{X}$ such that $\mathcal{D}(a) = a.\xi - \mu.a$, $(a \in \mathcal{A})$ [3]. $\mathcal{A}$ is called semi-amenable if every derivation $\mathcal{D} : \mathcal{A} \to \mathcal{X}^*$ is semi-inner.
We will sometimes abbreviate the phrase "bounded approximate identity" to B.A.I.
We recall that, for Banach algebras $\mathcal{A}$ and $\mathcal{B}$, the l_1-direct sum $\mathcal{A} \oplus \mathcal{B}$ equipped with the multiplication $(a, b).(c, d) := (ac, bd)$ is a Banach algebra.

2 Some notes on contractible and approximate contractible of Banach algebras

In this section, we will study some properties of contractible and approximate contractible Banach algebras which former have been studied for amenable case. We remark that a diagonal for a Banach algebra $\mathcal{A}$, is an element $d \in \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that,

1. $a\pi_{\mathcal{A}}(d) = a = \pi_{\mathcal{A}}(d)a$,
2. $a.d = d.a$,

for each $a \in \mathcal{A}$. Where $\pi_{\mathcal{A}} : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$ is the natural extension of the product map $a \otimes b \to ab$. For Banach algebras $\mathcal{A}$ and $\mathcal{B}$, the projective tensor product

$\mathcal{A}\widehat{\otimes}\mathcal{B}$, is the completion of $\mathcal{A}\otimes\mathcal{B}$, with respect to projective tensor norm and $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is a Banach algebra by canonical multiplication specified with

$$(a\otimes b).(c\otimes d) := (ac\otimes bd), \quad (a,c\in\mathcal{A} \ \textit{and} \ b,d\in\mathcal{B})$$

.

Theorem 1 *Let $\mathcal{A}$ be a Banach algebra. Then the following statements are equivalent.*

1. *$\mathcal{A}$ is contractible.*
2. *$\mathcal{A}$ is unital and has a diagonal.*
3. *$\mathcal{A}$ is semi-simple and finite dimensional.*

Proof. See [2] Theorem 1.9.21. □

In the following, we give some examples from contractible Banach algebras.

Example 1 (i) Suppose that $\mathcal{M}_n$, $(n\in\mathbb{N})$ is full matrix algebra of $n\times n$ complex matrices. Thus $\mathcal{M}_n$ is unital, and let $\{E_{j,i}: \ j,i=1,...,n\}$ be the set of canonical matrix units. Letting,

$$M := \frac{1}{n}\sum_{i,j=1}^{n} E_{j,i}\otimes E_{i,j}.$$

It's easily verified that M is a diagonal for $\mathcal{M}_n$, thus $\mathcal{M}_n$ is contractible by Theorem 1.
(ii) Let $l^p = l^p(\mathbb{N})$, $(1\le p<\infty)$ be Banach sequence algebra under pointwise operations. Set $E_n = \Sigma_{i=1}^{n}\delta_i$, $(n\in\mathbb{N})$ where δ_i is characteristic function of $\{i\}$ for $i\in\mathbb{N}$. We set $\mathcal{A}_n = E_n l^p$ $(n\in\mathbb{N})$, which is a finite-dimensional semi-simple subalgebra of l^p, and hence by Theorem 1 $\mathcal{A}_n$, $(n\in\mathbb{N})$ is contractible.

Proposition 1 *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\mathcal{I}$ be a closed ideal of $\mathcal{A}$.*

1. *If $\mathcal{B}$ is contractible and dense Banach subalgebra of $\mathcal{A}$, then $\mathcal{A}$ is contractible.*
2. *If $\mathcal{B}$ is contractible and $\theta:\mathcal{B}\to\mathcal{A}$ is a continuous homomorphism with dense range, then $\mathcal{A}$ is contractible.*
3. *If $\mathcal{I}$ and $\frac{\mathcal{A}}{\mathcal{I}}$ are contractible, then $\mathcal{A}$ is contractible.*
4. *If $\mathcal{A}$ and $\mathcal{B}$ are contractible, then $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is contractible.*

Proof. 1. Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and $\mathcal{D}\in\mathcal{Z}^1(\mathcal{A},\mathcal{X})$ be a continuous derivation. Since $\overline{\mathcal{B}}=\mathcal{A}$, so for each $a\in\mathcal{A}$, there is a net $(b_\alpha)_\alpha$ in $\mathcal{B}$ such that $b_\alpha\to a$ in norm topology of $\mathcal{A}$, thus for some $\mu\in\mathcal{X}$, we have

$$\mathcal{D}(a)=\lim_\alpha\mathcal{D}(b_\alpha)=\lim_\alpha(b_\alpha.\mu-\mu.b_\alpha)=a.\mu-\mu.a.$$

So $\mathcal{A}$ is contractible.

2. Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. Then $\mathcal{X}$ has $\mathcal{B}$-bimodule structure defined by $b.x := \theta(b)x, \quad x.b := x\theta(b)$. Now let $\mathcal{D} \in \mathcal{Z}^1(\mathcal{A}, \mathcal{X})$ be a continuous derivation. Thus $\mathcal{D}o\theta \in \mathcal{Z}^1(\mathcal{B}, \mathcal{X})$, and due to the contractibility of $\mathcal{B}$, is inner. So for some $\xi \in \mathcal{X}$, we have

$$\mathcal{D}o\theta(b) = b.\xi - \xi.b = \theta(b)\xi - \xi\theta(b).$$

Since for each $a \in \mathcal{A}$ there is a net $(b_\alpha)_\alpha$ in $\mathcal{B}$ which $\theta(b_\alpha) \to a$ in norm topology of $\mathcal{A}$. Then

$$\begin{aligned}\mathcal{D}(a) &= \lim_\alpha \mathcal{D}(\theta(b_\alpha))\\ &= \lim_\alpha (\mathcal{D} \circ \theta)(b_\alpha)\\ &= \lim_\alpha (b_\alpha.\xi - \xi.b_\alpha)\\ &= \lim_\alpha (\theta(b_\alpha)\xi - \xi\theta(b_\alpha))\\ &= a.\xi - \xi.a\end{aligned}$$

for each $a \in \mathcal{A}$. It follows that $\mathcal{D}$ is inner and $\mathcal{B}$ is contractible.
3. It is analogously arguing as in amenable case [[7] Proposition 5.1].
4. Since $\mathcal{A}$ and $\mathcal{B}$ are contractible, then by Theorem 1, there exist diagonals $M_\mathcal{A}$ and $M_\mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$, respectively. So

$$M_\mathcal{A} \otimes M_\mathcal{B} \in (\mathcal{A}\widehat{\otimes}\mathcal{A})\widehat{\otimes}(\mathcal{B}\widehat{\otimes}\mathcal{B}) \cong (\mathcal{A}\widehat{\otimes}\mathcal{B})\widehat{\otimes}(\mathcal{A}\widehat{\otimes}\mathcal{B})$$

is a diagonal of $\mathcal{A}\widehat{\otimes}\mathcal{B}$.

□

Theorem 2 *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. Then $\mathcal{A}$ and $\mathcal{B}$ are contractible if and only if $\mathcal{A} \oplus \mathcal{B}$ is contractible.*

Proof. Let $\mathcal{X}$ be a Banach $\mathcal{A}\oplus\mathcal{B}$-bimodule and $\mathcal{D} \in \mathcal{Z}^1(\mathcal{A}\oplus\mathcal{B}, \mathcal{X})$ be continuous. Then $\mathcal{X}$ is a Banach $\mathcal{A}$ (and $\mathcal{B}$)-bimodule by following module operations,

$$a.x := (a, 0)x, \quad x.a := x(a, 0),$$

$$(\text{and } b.x := (0, b)x, \ x.b := x(0, b)).$$

We define $\mathcal{D}_1 : \mathcal{A} \to \mathcal{X}^*$ with $\mathcal{D}_1(a) = \mathcal{D}(a, 0)$. Obviously, $\mathcal{D}_1$ is a continuous derivation. Thus according to contractibility of $\mathcal{A}$, $\mathcal{D}_1 = \delta_y$, for some $y \in \mathcal{X}$. So $\mathcal{D}^\sim = (\mathcal{D} - \delta_y) \in \mathcal{Z}^1(\mathcal{A} \oplus \mathcal{B}, \mathcal{X})$ and $\mathcal{D}^\sim|_{\mathcal{A}(\cong \mathcal{A}\oplus\{0_\mathcal{B}\})} = \{0\}$. Hence $\mathcal{D}^\sim : \mathcal{B}(\cong \{0_\mathcal{A}\} \oplus \mathcal{B}) \to \mathcal{X}$ is a continuous derivation. Therefore $\mathcal{D}^\sim = \delta_z$, for some $z \in \mathcal{X}$. Consequently, $\mathcal{D} = \delta_y + \delta_z = \delta_{y+z}$ and $\mathcal{A} \oplus \mathcal{B}$ is contractible.

Conversely, let $\mathcal{X}$ and $\mathcal{Y}$ be Banach $\mathcal{A}$-bimodule and $\mathcal{B}$-bimodule, respectively. Let $\mathcal{D}_1 \in \mathcal{Z}^1(\mathcal{A}, \mathcal{X})$ and $\mathcal{D}_2 \in \mathcal{Z}^1(\mathcal{B}, \mathcal{Y})$ be continuous. Then $\mathcal{X}\oplus\mathcal{Y}$ is a Banach $\mathcal{A} \oplus \mathcal{B}$-bimodule by following module actions

$$(a, b).(x, y) := (ax, by),$$

$$(x, y).(a, b) := (xa, yb).$$

The mapping

$$\mathcal{D} : \mathcal{A} \oplus \mathcal{B} \to \mathcal{X} \oplus \mathcal{Y},$$

$$\mathcal{D}(a, b) = (\mathcal{D}_1(a), \mathcal{D}_2(b)),$$

is a continuous derivation as follows,

$$\mathcal{D}[(a, b)(c, d)] = \mathcal{D}(ac, bd) = (\mathcal{D}_1(ac), \mathcal{D}_2(bd))$$

$$= (\mathcal{D}_1(a).c + a.\mathcal{D}_1(c), \mathcal{D}_2(b).d + b.\mathcal{D}_2(d)),$$

$$[\mathcal{D}(a, b)].(c, d) = (\mathcal{D}_1(a), \mathcal{D}_2(b)).(c, d)$$

$$= (\mathcal{D}_1(a).c, \mathcal{D}_2(b).d),$$

$$(a, b).[\mathcal{D}(c, d)] = (a, b).(\mathcal{D}_1(c), \mathcal{D}_2(d))$$

$$= (a.\mathcal{D}_1(c), b.\mathcal{D}_2(d)).$$

Thus, we have

$$\mathcal{D}[(a, b)(c, d)] = \mathcal{D}(a, b).(c, d) + (a, b).\mathcal{D}(c, d).$$

So by assumption $\mathcal{D}$ is inner. Then there is a $\psi = (x, y) \in \mathcal{X} \oplus \mathcal{Y}$ such that for each $(a, b) \in \mathcal{A} \oplus \mathcal{B}$,

$$(\mathcal{D}_1(a), \mathcal{D}_2(b)) = \mathcal{D}(a, b)$$

$$= (a, b).\psi - \psi.(a, b)$$

$$= (a, b).(x, y) - (x, y).(a, b)$$

$$= (ax - xa, by - yb).$$

Consequently, $\mathcal{D}_1$ and $\mathcal{D}_2$ are inner. Hence $\mathcal{A}$ and $\mathcal{B}$ are contractible. □

Theorem 3 *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. Suppose that $\mathcal{A} \oplus \mathcal{B}$ is approximately contractible, then so are $\mathcal{A}$ and $\mathcal{B}$.*

Proof. Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule, and $\mathcal{Y}$ be a Banach $\mathcal{B}$-bimodule, and $\mathcal{D}_1 \in \mathcal{Z}^1(\mathcal{A}, \mathcal{X})$ and $\mathcal{D}_2 \in \mathcal{Z}^1(\mathcal{B}, \mathcal{Y})$, be continuous derivations. Then $\mathcal{X} \oplus \mathcal{Y}$ is a Banach $\mathcal{A} \oplus \mathcal{B}$-bimodule by following multiplications,

$$(x, y).(a, b) := (xa, yb),$$

$$(a, b).(x, y) := (ax, by).$$

Now we define

$$\mathcal{D} : A \oplus B \to \mathcal{X} \oplus \mathcal{Y},$$

$$\mathcal{D}(a, b) = (\mathcal{D}_1(a), \mathcal{D}_2(b)).$$

$\mathcal{D}$ is a continuous derivation as follows,

$$\mathcal{D}[(a, b)(c, d)] = \mathcal{D}(ac, bd) = (\mathcal{D}_1(ac), \mathcal{D}_2(bd))$$

$$= (\mathcal{D}_1(a).c + a.\mathcal{D}_1(c), \mathcal{D}_2(b).d + b.\mathcal{D}_2(d))$$
$$= (\mathcal{D}_1(a).c, \mathcal{D}_2(b).d) + (a.\mathcal{D}_1(c), b.\mathcal{D}_2(d))$$
$$= (\mathcal{D}_1(a), \mathcal{D}_2(b)).(c, d) + (a, b).(\mathcal{D}_1(c), \mathcal{D}_2(d))$$
$$= \mathcal{D}(a, b).(c, d) + (a, b).\mathcal{D}(c, d).$$

So there exist a net $(\xi_\alpha)_\alpha$ in $\mathcal{X} \oplus \mathcal{Y}$, specified by $\xi_\alpha = (x_\alpha, y_\alpha)$, such that

$$(\mathcal{D}_1(a), \mathcal{D}_2(b)) =$$
$$\mathcal{D}(a, b) = \lim_\alpha((a, b).\xi_\alpha - \xi_\alpha.(a, b))$$
$$= \lim_\alpha(ax_\alpha - x_\alpha a, by_\alpha - y_\alpha b).$$

Hence $\mathcal{D}_1$ and $\mathcal{D}_2$ are approximately inner and $\mathcal{A}$, $\mathcal{B}$ are approximately contractible. □

Since approximately contractibility is equivalent approximately amenability [5, Theorem 2.1], so we have a similar version of this theorem for approximately amenability case which the proof is analogue almost verbatim.

Corollary 1 *If $\mathcal{A} \oplus \mathcal{A}$ is approximately contractible, then $\mathcal{A}$ has an approximate identity.*

Proof. It is an immediate consequence of Theorems 1 and 3. □

3 Connes amenability and Connes semi-amenability of Banach algebras

In 1972 Johnson, Kadison and Ringrose introduced a notion of amenability for von Neumann algebras in [9] which modified Johnson's original definition for amenability of Banach algebras in the sense that it takes the dual space structure of a von Neumann algebra into account. This notion of amenability was later called Connes-amenability by A. Ya. Helemskii in [6]. Later V. Runde extended the notion of Connes-amenability to the larger class of dual Banah algebras [11]. A Banach algebra $\mathcal{A}$ is said to be *dual* if there is a closed submodule $\mathcal{A}_*$ of $\mathcal{A}^*$, such that $\mathcal{A} = (\mathcal{A}_*)^*$. For a dual Banach algebra $\mathcal{A}$, a dual Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is called normal if, for each $x \in \mathcal{X}$ the maps $a \to a.x$ and $a \to x.a$ from $\mathcal{A}$ into $\mathcal{X}$, are w^*-continous. A dual Banach algebra $\mathcal{A}$ is Connes-amenable if, for every normal, dual Banach $\mathcal{A}$-bimodule $\mathcal{X}$, every w^*-continuous derivation $\mathcal{D} : \mathcal{A} \to \mathcal{X}$ is inner. We denote $\mathcal{Z}^1_{w^*}(\mathcal{A}, \mathcal{X})$ for the w^*-continuous derivations from $\mathcal{A}$ into $\mathcal{X}$ and $\mathcal{H}^1_{w^*}(\mathcal{A}, \mathcal{X}) = \mathcal{Z}^1_{w^*}(\mathcal{A}, \mathcal{X})/\mathcal{N}^1(\mathcal{A}, \mathcal{X})$.

Example 2 For a locally compact group G, $M(G)$ the Banach algebra of complex-valued, regular Borel measures on G, is dual Banach algebra with predual $C_0(G)$. Also if $\mathcal{A}$ is an Arens regular Banach algebra, then $\mathcal{A}^{**}$ is a dual Banach algebra with predual $\mathcal{A}^*$. All von Neumann algebras, and for reflexive Banach space $\mathcal{X}$, the Banach algebra of all bounded operators on $\mathcal{X}$, i.e. $B(\mathcal{X}) = (\mathcal{X} \widehat{\otimes} \mathcal{X}^*)^*$, are dual Banach algebras.

Definition 1 The dual Banach algebra $\mathcal{A}$ is *Connes semi-amenable* (resp. Connes approximately semi-amenable) if, every $\mathcal{D} \in \mathcal{Z}^1_{w^*}(\mathcal{A}, \mathcal{X})$, for each normal, dual Banach $\mathcal{A}$-bimodule $\mathcal{X}$, is semi-inner (resp. approximately semi-inner).

Lemma 1 *Let $\mathcal{A}$ be a dual Banach algebra. Suppose that $\mathcal{A}$ is Connes-amenable (or $Connes$ $semi-amenable$) then it is unital.*

Proof. By assumption $\mathcal{H}^1_{w^*}(\mathcal{A}, \mathcal{X}) = 0$, for every normal, dual Banach $\mathcal{A}$-bimodule $\mathcal{X}$. So the remainder of proof is analogous to argument of Proposition 4.1 in [10], almost verbatim. □

Proposition 2 *Let $\mathcal{A}$ be Banach algebra, and $\mathcal{B}$ be a dual Banach algebra, and let $\theta : \mathcal{A} \to \mathcal{B}$ be a continuous homomorphism with w^*-dense range. Then*

1. *If $\mathcal{A}$ is amenable, then $\mathcal{B}$ is Connes-amenable.*
2. *If $\mathcal{A}$ is dual and Connes-amenable, and if θ is w^*-continuous, then $\mathcal{B}$ is Connes-amenable.*

Proof. 1. Let $\mathcal{X}$ be a normal, dual Banach $\mathcal{B}$-bimodule and $\mathcal{D} \in \mathcal{Z}^1_{w_*}(\mathcal{B}, \mathcal{X})$. Then $\mathcal{X}$ is a normal, dual Banach $\mathcal{A}$-bimodule by following multiplication

$$x.a := x\theta(a), \quad a.x := \theta(a)x,$$

so $\mathcal{D}o\theta \in \mathcal{Z}^1_{w^*}(\mathcal{A}, \mathcal{X})$. By duality of $\mathcal{X}$, there exists $\mathcal{X}_* \subseteq \mathcal{X}^*$ a closed submodule of $\mathcal{X}^*$ such that $\mathcal{X} = (\mathcal{X}_*)^*$, now by amenability of $\mathcal{A}$ there exists $\xi \in \mathcal{X}$,

$$\mathcal{D} \circ \theta(a) = a.\xi - \xi.a = \theta(a)\xi - \xi\theta(a).$$

Since $\overline{\theta(\mathcal{A})}^{w^*} = \mathcal{B}$, so for each $b \in \mathcal{B}$ there exists a net $(a_\alpha)_\alpha$ in $\mathcal{A}$ such that $\theta(a_\alpha) \to b$ in w^*-topology of $\mathcal{B}$, and we have

$$\begin{aligned}\mathcal{D}(b) &= \mathcal{D}(w^* - \lim_\alpha \theta(a_\alpha)) \\ &= w^* - \lim_\alpha D(\theta(a_\alpha)) \\ &= w^* - \lim_\alpha (\theta(a_\alpha)\xi - \xi\theta(a_\alpha)) \\ &= b.\xi - \xi.b.\end{aligned}$$

Consequently, $\mathcal{D}$ is inner and $\mathcal{B}$ is Connes-amenable.

2. Let $\mathcal{X}$ be a normal, dual Banach $\mathcal{B}$-bimodule, and $\mathcal{D} \in \mathcal{Z}^1_{w^*}(\mathcal{B}, \mathcal{X})$. So by a similar argument as in (a), $\mathcal{X}$ is $\mathcal{A}$-bimodule and $\mathcal{D} \circ \theta \in \mathcal{Z}^1_{w^*}(\mathcal{A}, \mathcal{X})$. Therefore by assumption $\mathcal{D}o\theta$ and consequently $\mathcal{D}$ is inner. Thus $\mathcal{B}$ is Connes amenable.

□

Corollary 2 *Let $\mathcal{A}$ be a Arens regular Banach algebra. If $\mathcal{A}$ is amenable then $\mathcal{A}^{**}$ is Connes-amenable.*

Proof. Let $\pi : \mathcal{A} \to \mathcal{A}^{**}$ be canonical embedding, which is a continuous homomorphism. Firstly, $\mathcal{A}^{**}$ is a dual Banach algebra and according to *Goldstine's* Theorem $\overline{\pi(\mathcal{A})}^{w^*} = \mathcal{A}^{**}$. Therefore by Proposition 2 (a), $\mathcal{A}^{**}$ is Connes-amenable. □

Theorem 4 *Let $\mathcal{A}$ and $\mathcal{B}$ be dual Banach algebras. Then $\mathcal{A} \oplus \mathcal{B}$ is Connes-amenable $(resp.\ Connes\ semi-amenable)$ if and only if $\mathcal{A}$ and $\mathcal{B}$ are Connes-amenable $(resp.\ Connes\ semi-amenable)$.*

Proof. Let $\mathcal{X}$ and $\mathcal{Y}$ be normal, dual and Banach $\mathcal{A}$ and $\mathcal{B}$-bimodule, respectively. Let $\mathcal{D}_1 \in \mathcal{Z}^1_{w^*}(\mathcal{A}, \mathcal{X})$ and $\mathcal{D}_2 \in \mathcal{Z}^1_{w^*}(\mathcal{B}, \mathcal{Y})$. So $\mathcal{X} \oplus \mathcal{Y}$ is normal, dual and Banach $\mathcal{A} \oplus \mathcal{B}$-bimodule by module operations defined with

$$(a, b).(x, y) := (ax, by),\ (x, y).(a, b) := (xa, yb).$$

The mapping

$$\mathcal{D}^\sim : \mathcal{A} \oplus \mathcal{B} \to \mathcal{X} \oplus \mathcal{Y}$$
$$\mathcal{D}^\sim(a, b) = (\mathcal{D}_1(a), \mathcal{D}_2(b))$$

is a derivation, since

$$\begin{aligned}\mathcal{D}^\sim[(a, b)(c, d)] &= \mathcal{D}^\sim(ac, bd)\\ &= (\mathcal{D}_1(ac), \mathcal{D}_2(bd))\\ &= (\mathcal{D}_1(a).c + a.\mathcal{D}_1(c), \mathcal{D}_2(b).d + b.\mathcal{D}_2(d)).\end{aligned}$$

$$\begin{aligned}[\mathcal{D}^\sim(a, b)].(c, d) &= (\mathcal{D}_1(a), \mathcal{D}_2(b)).(c, d)\\ &= (\mathcal{D}_1(a).c, \mathcal{D}_2(b).d).\end{aligned}$$

$$\begin{aligned}(a, b).[\mathcal{D}^\sim(c, d)] &= (a, b).(\mathcal{D}_1(c), \mathcal{D}_2(d))\\ &= (a.\mathcal{D}_1(c), b.\mathcal{D}_2(d)).\end{aligned}$$

Thus,

$$\mathcal{D}^\sim[(a, b)(c, d)] = [\mathcal{D}^\sim(a, b)].(c, d) + (a, b).[\mathcal{D}^\sim(c, d)]$$

and by w^*-continuity of $\mathcal{D}_1$ and $\mathcal{D}_2$ we conclude $\mathcal{D}^\sim \in \mathcal{Z}^1_{w^*}(\mathcal{A} \oplus \mathcal{B}, \mathcal{X} \oplus \mathcal{Y})$. So due to the Connes-amenability $(resp.\ Connes\ semi-amenability)$ of $\mathcal{A} \oplus \mathcal{B}$, $\mathcal{D}^\sim$ is inner $(resp.\ semi-inner)$. Thus there is a $\psi = (x, y)$ $(resp.\ \mu = (x', y'),\ \eta = (x'', y''))$ in $X \oplus Y$ such that for each $(a, b) \in \mathcal{A} \oplus \mathcal{B}$ we have

$$\begin{aligned}(\mathcal{D}_1(a), \mathcal{D}_2(b)) &= \mathcal{D}^\sim(a, b)\\ &= (a, b).\psi - \psi.(a, b)\\ &= (a, b).(x, y) - (x, y).(a, b)\\ &= (ax - xa, by - yb).\end{aligned}$$

$$\begin{aligned}(resp.\ (\mathcal{D}_1(a), \mathcal{D}_2(b)) &= \mathcal{D}^{\sim}(a,b)\\ &= (a,b).\mu - \eta.(a,b)\\ &= (a,b).(x',y') - (x'',y'').(a,b)\\ &= (ax' - x''a, by' - y''b).)\end{aligned}$$

Consequently, $\mathcal{D}_1$ and $\mathcal{D}_2$ are inner ($resp.\ semi-inner$) and $\mathcal{A}$ and $\mathcal{B}$ are Connes-amenable ($resp.\ Connes\ semi-amenable$).
For the converse we argue Connes-amenability case and Connes semi-amenability holds similarly. Let $\mathcal{X}$ be a normal, dual Banach $\mathcal{A}\oplus\mathcal{B}$-bimodule and $\mathcal{D} \in \mathcal{Z}^1_{w^*}(\mathcal{A}\oplus\mathcal{B}, \mathcal{X})$. Then $\mathcal{X}$ is a normal, dual Banach $\mathcal{A}$ (and $\mathcal{B}$)-bimodule by following multiplications,

$$a.x := (a,0)x,\ x.a := x(a,0),$$

$$(\text{and } b.x := (0,b)x,\ x.b := x(0,b)).$$

We define $\mathcal{D}_1 : \mathcal{A} \to \mathcal{X}$ with $\mathcal{D}_1(a) = \mathcal{D}(a,0)$. Obviously, $\mathcal{D}_1$ is a w^*-continuous derivation. Thus according to Connes-amenability of $\mathcal{A}$, $\mathcal{D}_1 = \delta_y$, for some $y \in \mathcal{X}$. So $\mathcal{D}^{\sim} = (\mathcal{D}-\delta_y) \in \mathcal{Z}^1_{w^*}(\mathcal{A}\oplus\mathcal{B}, \mathcal{X})$ and $\mathcal{D}^{\sim}|_{\mathcal{A}} = \{0\}$. Hence $\mathcal{D}^{\sim} : \mathcal{B} \to \mathcal{X}$ is a w^*-continuous derivation. Therefore $\mathcal{D}^{\sim} = \delta_z$, for some $z \in \mathcal{X}$. Consequently, $\mathcal{D} = \delta_y + \delta_z = \delta_{y+z}$ and $\mathcal{A}\oplus\mathcal{B}$ is Connes-amenable. □

Proposition 3 *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are dual Banach algebras.*

1. *Let $\mathcal{B}$ be a dense Banach subalgebra of $\mathcal{A}$. If $\mathcal{B}$ is Connes semi-amenable then so is $\mathcal{A}$.*
2. *Let $\theta : \mathcal{B} \to \mathcal{A}$ be a continuous epimorphism. If $\mathcal{B}$ is Connes semi-amenable ($resp.\ approximately\ Connes\ semi-amenable$) then so is $\mathcal{A}$.*

Proof. Let $\mathcal{X}$ be a normal, dual Banach $\mathcal{A}$-bimodule and $\mathcal{D} \in \mathcal{Z}^1_{w^*}(\mathcal{A}, \mathcal{X})$.

1. Since $\overline{\mathcal{B}} = \mathcal{A}$, so for each $a \in \mathcal{A}$, there is a net $(b_\alpha)_\alpha$ in $\mathcal{B}$ such that $b_\alpha \to a$ in norm topology of $\mathcal{A}$. Also $\mathcal{D}|_{\mathcal{B}} \in \mathcal{Z}^1_{w^*}(\mathcal{B}, \mathcal{X})$, thus there exist $\mu, \eta \in \mathcal{X}$ such that, for each $a \in \mathcal{A}$ we have
$$\begin{aligned}\mathcal{D}(a) &= w^* - \lim_\alpha \mathcal{D}(b_\alpha)\\ &= w^* - \lim_\alpha (b_\alpha.\mu - \eta.b_\alpha)\\ &= a.\mu - \eta.a\end{aligned}$$
so $\mathcal{A}$ is Connes semi-amenable.
2. $\mathcal{X}$ is a normal, dual Banach $\mathcal{B}$-bimodule with actions induced via θ as follows,
$$b.x := \theta(b)x, \quad x.b := x\theta(b),$$
for $b \in \mathcal{B}$ and $x \in \mathcal{X}$. For $b, b' \in \mathcal{B}$ we have
$$\begin{aligned}\mathcal{D}\circ\theta(bb') &= \mathcal{D}(\theta(bb')) = \mathcal{D}(\theta(b).\theta(b'))\\ &= \mathcal{D}(\theta(b)).\theta(b') + \theta(b).\mathcal{D}(\theta(b'))\end{aligned}$$

$$= \mathcal{D} \circ \theta(b).b' + b.\mathcal{D} \circ \theta(b').$$

So $\mathcal{D} \circ \theta \in \mathcal{Z}^1_{w^*}(\mathcal{B}, \mathcal{X})$.
By assumption, $\mathcal{D} \circ \theta$ is semi-inner ($resp.\ approximately\ semi - inner$). Thus there are $\xi, \zeta \in \mathcal{X}$ (resp. nets $(\xi_\alpha)_\alpha$ and $(\zeta_\alpha)_\alpha$ in $\mathcal{X}$) such that for each $b \in \mathcal{B}$,

$$\mathcal{D} \circ \theta(b) = b.\xi - \zeta.b = \theta(b).\xi - \zeta.\theta(b).$$

$$\begin{aligned}(resp.\ \mathcal{D} \circ \theta(b) &= \lim_\alpha (b.\xi_\alpha - \zeta_\alpha.b) \\ &= \lim_\alpha (\theta(b).\xi_\alpha - \zeta_\alpha.\theta(b)).)\end{aligned}$$

Since $\theta(\mathcal{B}) = \mathcal{A}$, so for each $a \in \mathcal{A}$

$$\mathcal{D}(a) = a.\xi - \zeta.a.$$

$$(resp.\ \mathcal{D}(a) = \lim_\alpha (a.\xi_\alpha - \zeta_\alpha.a).)$$

Consequently, $\mathcal{A}$ is Connes semi-amenable (resp. approximately Connes semi-amenable).

□

Theorem 5 *Let $\mathcal{A}$ be a dual Banach algebra. Then $\mathcal{A}$ is Connes semi-amenable if and only if for each $\mathcal{D} \in \mathcal{Z}^1_{w^*}(\mathcal{A}, \mathcal{X})$ and normal, dual Banach $\mathcal{A}$-bimodule $\mathcal{X}$, there are bounded nets $(x_\alpha)_\alpha$ and $(y_\alpha)_\alpha$ in $\mathcal{X}$ such that, for all $a \in \mathcal{A}$, $\mathcal{D}(a) = \lim_\alpha (a.x_\alpha - y_\alpha.a) = \lim_\alpha \delta_{x_\alpha, y_\alpha}(a)$ in norm topology of $\mathcal{X}$.*

Proof. Let the dual Banach algebra $\mathcal{A}$ be Connes semi-amenable and $\mathcal{D} \in \mathcal{Z}^1_{w^*}(\mathcal{A}, \mathcal{X})$ for normal, dual Banach $\mathcal{A}$-bimodule $\mathcal{X}$. Since $\mathcal{X}^{**}$ has a dual $\mathcal{A}$-bimodule structure so $\mathcal{D}$ can be viewed as mapping into $\mathcal{X}^{**}$. By hypothesis there are $F, G \in \mathcal{X}^{**}$, such that $\mathcal{D}(a) = a.F - G.a$. Let $(x_\alpha)_\alpha$ and $(y_\alpha)_\alpha$ be bounded nets in $\mathcal{X}$, which $\widehat{x_\alpha} \to F$ and $\widehat{y_\alpha} \to G$ in w^*-topology on $\mathcal{X}^{**}$. For $f \in \mathcal{X}^*$ we have

$$\begin{aligned}\langle f, \mathcal{D}(a) \rangle &= \langle f, a.F - G.a \rangle \\ &= \langle f.a, F \rangle - \langle a.f, G \rangle \\ &= \lim_\alpha \langle f.a, \widehat{x_\alpha} \rangle - \lim_\alpha \langle a.f, \widehat{y_\alpha} \rangle \\ &= \lim_\alpha \langle x_\alpha, f.a \rangle - \lim_\alpha \langle y_\alpha, a.f \rangle \\ &= \lim_\alpha \langle a.x_\alpha, f \rangle - \lim_\alpha \langle y_\alpha.a, f \rangle \\ &= \lim_\alpha \langle a.x_\alpha - y_\alpha.a, f \rangle.\end{aligned}$$

So $\mathcal{D}(a) = w - \lim_\alpha (a.x_\alpha - y_\alpha.a)$ in weak topology of $\mathcal{X}$. Now by passing to convex combination and using the fact that the weak closure of a convex set is the same as it's norm closure, then we have $\mathcal{D}(a) = \lim_\alpha (a.x_\alpha - y_\alpha.a) = \lim_\alpha \delta_{x_\alpha, y_\alpha}(a)$.

Conversely, let $\mathcal{D}$ be a w^*-continuous derivation from $\mathcal{A}$ into a normal, dual

Banach $\mathcal{A}$-bimodule $\mathcal{X}$. So by assumption there are bounded nets $(x_\alpha)_\alpha$ and $(y_\alpha)_\alpha$ in $\mathcal{X}$, such that $\mathcal{D}(a) = \lim_\alpha(a.x_\alpha - y_\alpha.a)$ in the norm topology of $\mathcal{X}$. Let x_0 and y_0 be the w^*-cluster points of $(x_\alpha)_\alpha$ and $(y_\alpha)_\alpha$ in $\mathcal{X}$, respectively. By w^*-continuity of the module action in a normal, dual $\mathcal{A}$-bimodule, we then get $\mathcal{D}(a) = a.x_0 - y_0.a$, for all $a \in \mathcal{A}$. Therefore $\mathcal{D}$ is semi-inner and $\mathcal{A}$ is Connes semi-amenable. □

Theorem 6 *[11] Let $\mathcal{A}$ be an Arens regular Banach algebra which is an ideal in $\mathcal{A}^{**}$. Then $\mathcal{A}$ is semi-amenable if and only if $\mathcal{A}^{**}$ is Connes semi-amenable.*

Proof. Let $\mathcal{A}$ be semi-amenable, $\mathcal{X}$ be a normal, dual Banach $\mathcal{A}^{**}$-bimodule and $\mathcal{D} \in \mathcal{Z}^1_{w^*}(\mathcal{A}^{**}, \mathcal{X})$. Let $\mathcal{D}_1$ be the restriction of $\mathcal{D}$ on $\mathcal{A}$. So $\mathcal{D}_1 \in Z^1_{w^*}(\mathcal{A}, \mathcal{X})$ and by assumption $\mathcal{D}_1$ is semi-inner. Thus there are $\mu, \eta \in \mathcal{X}$ such that, for each $a \in \mathcal{A}$,

$$\mathcal{D}_1(a) = a.\mu - \eta.a.$$

Let $F \in \mathcal{A}^{**}$, so by Goldstine's Theorem there exists a bounded net $(a_\alpha)_\alpha$ in $\mathcal{A}$ such that $a_\alpha \to F$ in w^*-topology of $\mathcal{A}^{**}$. Then by w^*-continuity of module actions we have

$$\begin{aligned}\mathcal{D}(F) &= w^* - \lim_\alpha \mathcal{D}(a_\alpha) \\ &= w^* - \lim_\alpha \mathcal{D}_1(a_\alpha) \\ &= w^* - \lim_\alpha (a_\alpha.\mu - \eta.a_\alpha) \\ &= F.\mu - \eta.F.\end{aligned}$$

Therefore $\mathcal{A}^{**}$ is Connes semi-amenable.

Conversely, let $\mathcal{A}^{**}$ be Connes semi-amenable. Then by Lemma 1, $\mathcal{A}^{**}$ has an identity element. Consequently, $\mathcal{A}$ as a closed two-sided ideal of $\mathcal{A}^{**}$ has a bounded approximate identity, say $(e_\alpha)_\alpha$ in $\mathcal{A}$. The remainder of proof is verbatim analogue of Theorem 4.4.8 in [11]. □

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