Fixed Points of $(\psi, \varphi)_{\Omega}$-Contractive Mappings in Ordered P-Metric Spaces

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Abstract In this paper, we introduce the notion of an extended metric space (p-metric space) as a new generalization of the concept of b-metric space. Also, we present the concept of $(\psi, \varphi)_{\Omega}$-contractive mappings and we establish some fixed point results for this class of mappings in ordered complete p-metric spaces. Our results generalize several well-known comparable results in the literature. Finally, examples support our results.

Keywords Fixed point · Complete metric space · Ordered b-metric space

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1 Introduction

The Banach contraction principle [5] is a very powerful tool for solving problems in nonlinear analysis. Some authors generalized this interesting theorem in different ways (see, e.g., [1,2,7,8,10,11,14,18]).

Khan et al. [17] introduced the concept of an altering distance function as follows.

Definition 1 [17] The function $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ is called an altering distance function, if the following properties hold:

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1. \( \varphi \) is continuous and non-decreasing.
2. \( \varphi(t) = 0 \) if and only if \( t = 0 \).

So far, many authors have studied fixed point theorems which are based on altering distance functions (see, e.g., \([1,12,15,17,19,21,23,24]\)).

The concept of a \( b \)-metric space was introduced by Czerwik in \([9]\). After that, several interesting results about the existence of a fixed point for single-valued and multi-valued operators in \( b \)-metric spaces have been obtained (see, \([1,3,4,6,13,16,20,22,26]\)).

**Definition 2** \([9]\) Let \( X \) be a (nonempty) set and \( s \geq 1 \) be a given real number. A function \( d : X \times X \to \mathbb{R}^+ \) is a \( b \)-metric iff for all \( x,y,z \in X \), the following conditions hold:

- \((b_1)\) \( d(x,y) = 0 \) iff \( x = y \),
- \((b_2)\) \( d(x,y) = d(y,x) \),
- \((b_3)\) \( d(x,z) \leq s[d(x,y) + d(y,z)] \).

In this case, the pair \((X,d)\) is called a \( b \)-metric space.

A \( b \)-metric is a metric, when \( s = 1 \).

Motivated with \([9]\), the following definitions and results will be needed in the sequel.

**Definition 3** Let \( X \) be a (nonempty) set. A function \( \tilde{d} : X \times X \to \mathbb{R}^+ \) is a \( p \)-metric iff for all \( x,y,z \in X \), the following conditions hold:

- \((p_1)\) \( \tilde{d}(x,y) = 0 \) iff \( x = y \),
- \((p_2)\) \( \tilde{d}(x,y) = \tilde{d}(y,x) \),
- \((p_3)\) \( \tilde{d}(x,z) \leq \Omega(\tilde{d}(x,y) + \tilde{d}(y,z)) \).

In this case, the pair \((X,\tilde{d})\) is called a \( p \)-metric space, or, an extended \( b \)-metric space.

It should be noted that, the class of \( p \)-metric spaces is considerably larger than the class of \( b \)-metric spaces, since a \( b \)-metric is a \( p \)-metric, when \( \Omega(x) = sx \) while a metric is a \( p \)-metric, when \( \Omega(x) = x \).

Here, we present an example to show that in general, a \( p \)-metric need not necessarily to be a \( b \)-metric.

**Example 1** Let \((X,\tilde{d})\) be a metric space and \( \rho(x,y) = \sinh \tilde{d}(x,y) \). We show that \( \rho \) is a \( p \)-metric with \( \Omega(t) = \sinh(t) \) for all \( t \geq 0 \).

Obviously, conditions \((p_1)\) and \((p_2)\) of Definition 3 are satisfied.
For each $x, y, z \in X$,
\[
\rho(x, y) = \sinh \left( e^{\tilde{d}(x, y)} \right) \leq \sinh \left( e^{\tilde{d}(x, z) + \tilde{d}(z, y)} \right) \\
\leq \sinh \left[ \sinh \left( e^{\tilde{d}(x, z)} \right) + \sinh \left( e^{\tilde{d}(z, y)} \right) \right] \leq \sinh \left( e^{\rho(x, z) + \rho(z, y)} \right) = \Omega(\rho(x, z) + \rho(z, y)).
\]

So, condition $(p_3)$ of Definition 3 is also satisfied and $\rho$ is a $p$-metric. Note that, $\sinh|x - y|$ is not a metric on $\mathbb{R}$, as we know that
\[
\sinh 5 = 74.2032105778 \geq 3.62686040785 + 10.0178749274 = \sinh 2 + \sinh 3.
\]
Obviously, $\sinh|x - y|$ is not also a $b$-metric for any $s \geq 1$.

**Example 2** Let $(X, \tilde{d})$ be a metric space and $\rho(x, y) = e^{\tilde{d}(x, y)} - 1$. We show that $\rho$ is a $p$-metric with $\Omega(t) = e^t - 1$.

Obviously, conditions $(p_1)$ and $(p_2)$ of Definition 3 are satisfied.

On the other hand, for each $x, y, z \in X$,
\[
\rho(x, y) = e^{\tilde{d}(x, y)} - 1 \leq e^{\tilde{d}(x, z) + \tilde{d}(z, y)} - 1 \\
\leq e^{\tilde{d}(x, z)} + e^{\tilde{d}(z, y)} - 1 - 1 \\
= e^{\rho(x, z) + \rho(z, y)} - 1 \\
= \Omega(\rho(x, z) + \rho(z, y)).
\]

So, condition $(p_3)$ of Definition 3 is also satisfied and $\rho$ is a $p$-metric.

In general, we have the following proposition.

**Proposition 1** Let $(X, \tilde{d})$ be a metric space with coefficient $s \geq 1$ and let $\rho(x, y) = \xi(\tilde{d}(x, y))$ where $\xi : [0, \infty) \rightarrow [0, \infty)$ is a strictly increasing function with $x \leq \xi(x)$ and $0 = \xi(0)$. We show that $\rho$ is a $p$-metric with $\Omega(t) = \xi(t)$.

For each $x, y, z \in X$,
\[
\rho(x, y) = \xi(\tilde{d}(x, y)) \leq \xi(\tilde{d}(x, z) + \tilde{d}(z, y)) \\
\leq \xi(\xi(\tilde{d}(x, z)) + \xi(\tilde{d}(z, y))) = \Omega(\rho(x, z) + \rho(z, y)).
\]

So, $\rho$ is a $p$-metric.

The above proposition constructs the following example:

**Example 3** Let $(X, \tilde{d})$ be a metric space and let $\rho(x, y) = e^{\tilde{d}(x, y)} \sec^{-1}(e^{\tilde{d}(x, y)})$. Then $\rho$ is a $p$-metric with $\Omega(t) = e^t \sec^{-1}(e^t)$.
Definition 4 Let \((X, \tilde{d})\) be a \(p\)-metric space. Then a sequence \(\{x_n\}\) in \(X\) is called:

(a) \(p\)-convergent if and only if there exists \(x \in X\) such that \(\tilde{d}(x_n, x) \to 0\), as \(n \to +\infty\). In this case, we write \(\lim_{n \to \infty} x_n = x\).

(b) \(p\)-Cauchy if and only if \(\tilde{d}(x_n, x_m) \to 0\) as \(n, m \to +\infty\).

(c) The \(p\)-metric space \((X, \tilde{d})\) is \(p\)-complete if every \(p\)-Cauchy sequence in \(X\) \(p\)-converges.

Proposition 2 In a \(p\)-metric space \((X, \tilde{d})\), as \(\Omega(0) = 0\),

1. \(p\)-convergent sequence has a unique limit.

2. Each \(p\)-convergent sequence is \(p\)-Cauchy.

3. In general, a \(p\)-metric is not continuous.

We will need the following simple lemma about the \(p\)-convergent sequences.

Lemma 1 Let \((X, \tilde{d})\) be a \(p\)-metric space with a strictly increasing continuous function \(\Omega : [0, \infty) \to [0, \infty)\), and suppose that \(\{x_n\}\) and \(\{y_n\}\) \(p\)-converge to \(x, y\), respectively. Then, we have

\[
(\Omega^2)^{-1}(\tilde{d}(x, y)) \leq \liminf_{n \to \infty} \tilde{d}(x_n, y_n) \leq \limsup_{n \to \infty} \tilde{d}(x_n, y_n) \leq \Omega(\tilde{d}(x, y)).
\]

In particular, if \(x = y\), then, \(\lim_{n \to \infty} \tilde{d}(x_n, y_n) = 0\). Moreover, for each \(z \in X\) we have

\[
\Omega^{-1}(\tilde{d}(x, z)) \leq \liminf_{n \to \infty} \tilde{d}(x_n, z) \leq \limsup_{n \to \infty} \tilde{d}(x_n, z) \leq \Omega(\tilde{d}(x, z)).
\]

Proof. (a) Using the \(p\)-triangular inequality, it is easy to see that

\[
\tilde{d}(x, y) \leq \Omega(\tilde{d}(x, x_n) + \tilde{d}(x_n, y)) \\
\leq \Omega(\tilde{d}(x, x_n) + \Omega(\tilde{d}(x_n, y_n) + \tilde{d}(y_n, y)))
\]

and

\[
\tilde{d}(x_n, y_n) \leq \Omega(\tilde{d}(x_n, x) + \Omega(\tilde{d}(x, y) + \tilde{d}(y, y_n))).
\]

Taking the lower limit as \(n \to \infty\) in the first inequality one has

\[
\tilde{d}(x, y) \leq \Omega(\Omega(\liminf_{n \to \infty} \tilde{d}(x_n, y_n)))
\]

and taking the upper limit as \(n \to \infty\) in the second inequality we have

\[
\limsup_{n \to \infty} \tilde{d}(x_n, y_n) \leq \Omega(\Omega(\tilde{d}(x, y))).
\]

(b) Using the \(p\)-triangular inequality, it is easy to see that

\[
\tilde{d}(x, z) \leq \Omega(\tilde{d}(x, x_n) + \tilde{d}(x_n, z))
\]

and

\[
\tilde{d}(x_n, z) \leq \Omega(\tilde{d}(x_n, x) + \tilde{d}(x, z)).
\]
Taking the lower limit as \( n \to \infty \) in the first inequality one has
\[
\hat{d}(x, z) \leq \Omega(\liminf_{n \to \infty} \tilde{d}(x_n, z))
\]
and taking the upper limit as \( n \to \infty \) in the second inequality we have
\[
\limsup_{n \to \infty} \tilde{d}(x_n, z) \leq \Omega(\hat{d}(x, z)).
\]

In this paper, we introduce the notion of generalized \( (\psi, \varphi)_{\Omega} \)-contractive mapping and we establish some results in complete ordered \( p \)-metric spaces, where \( \psi \) and \( \varphi \) are altering distance functions. Our results generalize several comparable results in the literature.

## 2 Main results

In this section, we define the notion of \( (\psi, \varphi)_{\Omega} \)-contractive mapping and prove our new results.

Let \( (X, \preceq, d) \) be an ordered \( p \)-metric space and let \( f : X \to X \) be a mapping. Set
\[
M(x, y) = \max \left\{ \tilde{d}(x, y), \tilde{d}(x, fx), \tilde{d}(y, fy), \tilde{d}(y, fx) \right\}.
\]  

**Definition 5** Let \( (X, \preceq, d) \) be an ordered \( p \)-metric space. We say that a mapping \( f : X \to X \) is an ordered \( (\psi, \varphi)_{\Omega} \)-contractive mapping if there exist two altering distance functions \( \psi \) and \( \varphi \) and strictly increasing continuous function \( \Omega : [0, \infty) \to [0, \infty) \) with \( x \leq \Omega(x) \), for all nonnegative real number \( x \) such that
\[
\psi(\Omega(\tilde{d}(fx, fy))) \leq \psi(M(x, y)) - \varphi(M(x, y))
\]
for all comparable elements \( x, y \in X \).

Now, let us to prove our first result.

**Theorem 1** Let \( (X, \preceq, d) \) be a partially ordered \( p \)-complete \( p \)-metric space. Let \( f : X \to X \) be an ordered non-decreasing continuous ordered \( (\psi, \varphi)_{\Omega} \)-contractive mapping. If there exists \( x_0 \in X \) such that \( x_0 \preceq fx_0 \), then \( f \) has a fixed point.

**Proof:** Let \( x_0 \in X \) be arbitrary. Define a sequence \( (x_n) \) in \( X \) such that \( x_{n+1} = fx_n \) for all \( n \geq 0 \). Since \( x_0 \preceq fx_0 = x_1 \) and \( f \) is non-decreasing, we have \( x_1 = fx_0 \preceq x_2 = fx_1 \). Inductively, we have
\[
x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots.
\]
If \( x_n = x_{n+1} \), for some \( n \in \mathcal{N} \), then \( x_n = f x_n \) and hence \( x_n \) is a fixed point of \( f \). So, we may assume that \( x_n \neq x_{n+1} \), for all \( n \in \mathcal{N} \). By (6), we have

\[
\psi(\tilde{d}(x_n, x_{n+1})) \leq \psi(\Omega(\tilde{d}(x_n, x_{n+1}))) = \psi(\Omega(d(x_{n-1}, fx_n))) \leq \psi(M(x_{n-1}, x_n)) - \varphi(M(x_{n-1}, x_n)),
\]

(7)

where

\[
M(x_{n-1}, x_n) = \max \left\{ \tilde{d}(x_{n-1}, x_n), \tilde{d}(x_{n-1}, fx_{n-1}), \tilde{d}(x_n, fx_n), \tilde{d}(x_n, f x_{n-1}) \right\}
\]

\[
= \max \left\{ \tilde{d}(x_{n-1}, x_n), \tilde{d}(x_n, x_{n+1}) \right\}
\]

\[
= \max \left\{ \tilde{d}(x_{n-1}, x_n), \tilde{d}(x_n, x_{n+1}) \right\}.
\]

(8)

From (7) and (8) and the properties of \( \psi \) and \( \varphi \), we get

\[
\psi(\tilde{d}(x_n, x_{n+1})) \leq \psi \left( \max \left\{ \tilde{d}(x_{n-1}, x_n), \tilde{d}(x_n, x_{n+1}) \right\} \right) - \varphi \left( \max \left\{ \tilde{d}(x_{n-1}, x_n), \tilde{d}(x_n, x_{n+1}) \right\} \right) < \psi(\tilde{d}(x_n, x_{n+1})).
\]

(9)

If

\[
\max \left\{ \tilde{d}(x_{n-1}, x_n), \tilde{d}(x_n, x_{n+1}) \right\} = \tilde{d}(x_n, x_{n+1}),
\]

then by (9) we have

\[
\psi(\tilde{d}(x_n, x_{n+1})) \leq \psi(\tilde{d}(x_n, x_{n+1})) - \varphi(\tilde{d}(x_n, x_{n+1})) < \psi(\tilde{d}(x_n, x_{n+1})),
\]

(10)

which gives a contradiction. Thus,

\[
\max \left\{ \tilde{d}(x_{n-1}, x_n), \tilde{d}(x_n, x_{n+1}) \right\} = \tilde{d}(x_n, x_{n+1}).
\]

(11)

Therefore (9) becomes

\[
\psi(\tilde{d}(x_n, x_{n+1})) \leq \psi(\tilde{d}(x_n, x_{n+1})) - \varphi(\tilde{d}(x_n, x_{n+1})) < \psi(\tilde{d}(x_n, x_{n+1})).
\]

(12)

Since \( \psi \) is a non-decreasing mapping, \( \{\tilde{d}(x_n, x_{n+1}) : n \in \mathcal{N} \cup \{0\} \} \) is a non-increasing sequence of positive numbers. So, there exists \( r \geq 0 \) such that

\[
\lim_{n \to \infty} \tilde{d}(x_n, x_{n+1}) = r.
\]

(13)

Letting \( n \to \infty \) in (11), we get

\[
\psi(r) \leq \psi(r) - \varphi(r) \leq \psi(r).
\]

(14)
Therefore, \( \varphi(r) = 0 \), and hence \( r = 0 \). Thus, we have
\[
\lim_{n \to \infty} \tilde{d}(x_n, x_{n+1}) = 0. \tag{14}
\]

Next, we show that \( \{x_n\} \) is a \( p \)-Cauchy sequence in \( X \). By contradiction, there exists \( \varepsilon > 0 \) for which we can find two subsequences \( \{x_{m_i}\} \) and \( \{x_{n_i}\} \) of \( \{x_n\} \) such that \( n_i \) is the smallest index for which
\[
 n_i > m_i > i, \quad \tilde{d}(x_{m_i}, x_{n_i}) \geq \varepsilon. \tag{15}
\]
This means that
\[
\tilde{d}(x_{m_i}, x_{n_i-1}) < \varepsilon. \tag{16}
\]

From (15) and using the \( p \)-triangular inequality, we get
\[
\varepsilon \leq \tilde{d}(x_{m_i}, x_{n_i-1}) \leq \Omega(\tilde{d}(x_{m_i}, x_{m_i-1}) + \tilde{d}(x_{m_i-1}, x_{n_i})), \tag{17}
\]
\[
\leq \Omega(\tilde{d}(x_{m_i}, x_{m_i-1}) + \tilde{d}(x_{m_i-1}, x_{n_i-1}) + \tilde{d}(x_{n_i-1}, x_{n_i}))).
\]
Using (17) and taking the upper limit as \( i \to \infty \), we get
\[
(\Omega^2)^{-1}(\varepsilon) \leq \liminf_{i \to \infty} \tilde{d}(x_{m_i-1}, x_{n_i-1}). \tag{18}
\]
On the other hand, we have
\[
\tilde{d}(x_{m_i-1}, x_{n_i-1}) \leq \Omega(\tilde{d}(x_{m_i-1}, x_{m_i}) + \tilde{d}(x_{m_i}, x_{n_i})). \tag{19}
\]
Using (14), (16) and taking the upper limit as \( i \to \infty \), we get
\[
\limsup_{i \to \infty} \tilde{d}(x_{m_i-1}, x_{n_i-1}) \leq \Omega(\varepsilon). \tag{20}
\]
On the other hand, we have
\[
\tilde{d}(x_{m_i}, x_{n_i}) \leq \Omega(\tilde{d}(x_{m_i}, x_{m_i-1}) + \tilde{d}(x_{m_i-1}, x_{n_i})). \tag{21}
\]
Using (14), (15) and taking the upper limit as \( i \to \infty \), we get
\[
\limsup_{i \to \infty} \tilde{d}(x_{m_i}, x_{n_i-1}) \geq \Omega^{-1}(\varepsilon). \tag{22}
\]
From (6), we have
\[
\psi(\Omega(\tilde{d}(x_{m_i}, x_{n_i}))) = \psi(\Omega(\tilde{d}(fx_{m_i-1}, fx_{n_i-1}))) \leq \psi(M(x_{m_i-1}, x_{n_i-1})) - \varphi(M(x_{m_i-1}, x_{n_i-1})), \tag{23}
\]
where
\[
M(x_{m_i-1}, x_{n_i-1}) = \max \left\{ \tilde{d}(x_{m_i-1}, x_{n_i-1}), \tilde{d}(x_{m_i-1}, fx_{m_i-1}), \tilde{d}(x_{n_i-1}, fx_{n_i-1}), \tilde{d}(x_{n_i-1}, fx_{m_i-1}) \right\} \tag{24}
\]
Taking the upper limit as $i \to \infty$ in (24) and using (14), we get

$$\limsup_{i \to \infty} M(x_{m_{i-1}}, x_{n_{i-1}}) = \max\{\limsup_{i \to \infty} \tilde{d}(x_{m_{i-1}}, x_{n_{i-1}}), 0, 0, \limsup_{i \to \infty} \tilde{d}(x_{m_{i}}, x_{n_{i-1}})\} \leq \Omega(\varepsilon).$$

(25)

So, we have

$$\limsup_{i \to \infty} M(x_{m_{i-1}}, x_{n_{i-1}}) \leq \Omega(\varepsilon),$$

(26)

Similarly, we obtain that

$$(\Omega^2)^{-1}(\varepsilon) \leq \liminf_{i \to \infty} M(x_{m_{i-1}}, x_{n_{i-1}}).$$

(27)

Now, taking the upper limit as $i \to \infty$ in (23) and using (26) and (27), we have

$$\psi(\Omega(\varepsilon)) \leq \psi(\Omega(\limsup_{i \to \infty} \tilde{d}(x_{m_{i}}, x_{n_{i}})))$$

$$\leq \psi(\limsup_{i \to \infty} M(x_{m_{i-1}}, x_{n_{i-1}})) - \liminf_{i \to \infty} \varphi(M(x_{m_{i-1}}, x_{n_{i-1}}))$$

$$\leq \psi(\Omega(\varepsilon)) - \varphi(\liminf_{i \to \infty} M(x_{m_{i-1}}, x_{n_{i-1}})).$$

(28)

which further implies that

$$\varphi(\liminf_{i \to \infty} M(x_{m_{i-1}}, x_{n_{i-1}})) = 0,$$

so $\liminf_{i \to \infty} M(x_{m_{i-1}}, x_{n_{i-1}}) = 0$, a contradiction to (27). Thus, $\{x_{n+1} = fx_n\}$ is a $p$-Cauchy sequence in $X$. As $X$ is a $p$-complete space, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$, and

$$\lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} fx_n = u.$$

Now, as $f$ is continuous, using the $p$-triangular inequality, we get

$$\tilde{d}(u, fu) \leq \Omega(\tilde{d}(u, fx_n) + \tilde{d}(fx_n, fu)).$$

Letting $n \to \infty$, we get

$$\tilde{d}(u, fu) \leq \Omega(\lim_{n \to \infty} \tilde{d}(u, fx_n) + \lim_{n \to \infty} \tilde{d}(fx_n, fu)) = 0.$$

So, we have $fu = u$. Thus, $u$ is a fixed point of $f$.

Note that the continuity of $f$ in Theorem 1 is not necessary and can be dropped.

Recall that, an ordered $p$-metric space $(X, \preceq, p)$ is said to have sequential limit comparison property (s.l.c.p) if for every nondecreasing sequence $\{x_n\}$ in $X$, converging to some $x \in X$, $x_n \preceq x$ holds for all $n \in \mathbb{N}$.

**Theorem 2** Under the same hypotheses of Theorem 1, without the continuity assumption of $f$, assume that $(X, \preceq, p)$ enjoys the s.l.c.p. Then $f$ has a fixed point in $X$. 
**Proof.** Following the proof of Theorem 1, we construct an increasing sequence \( \{x_n\} \) in \( X \) such that \( x_n \to u \), for some \( u \in X \). Using the assumption s.l.c.p. on \( X \), we have \( x_n \preceq u \), for all \( n \in \mathbb{N} \). Now, we show that \( fu = u \). By (6), we have

\[
\psi(\Omega(\tilde{d}(x_{n+1}, fu))) = \psi(\Omega(\tilde{d}(fx_n, fu))) \\
\leq \psi(M(x_n, u)) - \varphi(M(x_n, u)),
\]

where

\[
M(x_n, u) = \max \{ \tilde{d}(x_n, u), \tilde{d}(x_n, fx_n), \tilde{d}(u, fu), \tilde{d}(fx_n, u) \} \\
= \max \{ \tilde{d}(x_n, u), \tilde{d}(x_n, x_{n+1}), \tilde{d}(u, fu), \tilde{d}(x_{n+1}, u) \}.
\]

Letting \( n \to \infty \) in (30) and using Lemma 1, we get

\[
\limsup_{i \to \infty} M(x_n, u) = \tilde{d}(u, fu).
\]

Similarly, we can obtain

\[
\liminf_{i \to \infty} M(x_n, u) = \tilde{d}(u, fu).
\]

Again, taking the upper limit as \( n \to \infty \) in (29) and using Lemma 1 and (31) we get

\[
\psi(\tilde{d}(u, fu)) = \psi(\Omega(\tilde{d}(x_{n+1}, fu))) \\
\leq \psi(\limsup_{i \to \infty} M(x_n, u)) - \liminf_{i \to \infty} \varphi(M(x_n, u)) \\
\leq \psi(\tilde{d}(u, fu)) - \varphi(\liminf_{i \to \infty} M(x_n, u)).
\]

Therefore, \( \varphi(\liminf_{n \to \infty} M(x_n, u)) \leq 0 \), equivalently, \( \liminf_{n \to \infty} M(x_n, u) = 0 \). Thus, from (32) we get \( u = fu \) and hence \( u \) is a fixed point of \( f \).

**Corollary 1** Let \((X, \preceq, \tilde{d})\) be a partially ordered \( p \)-complete \( p \)-metric space. Let \( f : X \to X \) be an ordered non-decreasing mapping. Suppose that there exist \( k \in [0, 1) \) such that

\[
\Omega(\tilde{d}(fx, fy)) \leq k \max \{ \tilde{d}(x, y), \tilde{d}(x, fx), \tilde{d}(y, fy), \tilde{d}(y, fx) \},
\]

for all comparable elements \( x, y \in X \). If there exists \( x_0 \in X \) such that \( x_0 \preceq fx_0 \), then \( f \) has a fixed point provided that \( f \) is continuous, or, \((X, \preceq, p)\) enjoys the s.l.c.p.

**Proof.** Follows from Theorems (1) and (2) and by taking \( \psi(t) = t \) and \( \varphi(t) = (1 - k)t \), for all \( t \in [0, +\infty) \).
Corollary 2 Let \((X, \preceq, d)\) be a partially ordered \(p\)-complete \(p\)-metric space. Let \(f : X \to X\) be an ordered non-decreasing mapping. Suppose that there exist \(\alpha, \beta, \gamma, \delta \in [0, 1)\) with \(\alpha + \beta + \gamma + \delta \in [0, 1)\) such that

\[
\Omega(\ddot{d}(fx, fy)) \leq \alpha \ddot{d}(x, y) + \beta \ddot{d}(x, fx) + \gamma \ddot{d}(y, fy) + \delta \ddot{d}(y, fx),
\]

for all comparable elements \(x, y \in X\). If there exists \(x_0 \in X\) such that \(x_0 \preceq fx_0\), then \(f\) has a fixed point provided that \(f\) is continuous, or, \((X, \preceq, p)\) enjoys the s.l.c.p.

The following corollary is an extension of Banach contraction principle in an extended \(b\)-metric space.

Corollary 3 Let \((X, \preceq, d)\) be a partially ordered \(p\)-complete \(p\)-metric space. Let \(f : X \to X\) be an ordered non-decreasing mapping. Suppose that there exist \(\alpha \in [0, 1)\) such that

\[
\sinh(\ddot{d}(fx, fy)) \leq \alpha \ddot{d}(x, y),
\]

for all comparable elements \(x, y \in X\). If there exists \(x_0 \in X\) such that \(x_0 \preceq fx_0\), then \(f\) has a fixed point provided that \(f\) is continuous, or, \((X, \preceq, p)\) enjoys the s.l.c.p.

Now, in order to support the usability of our results, we present the following examples.

Example 4 Let \(X = [0, 100]\) be equipped with the \(p\)-metric \(\ddot{d}(x, y) = e^{|x-y|^2} - 1\) for all \(x, y \in X\), where \(\Omega(x) = e^{2x} - 1\).

Define a relation \(\preceq\) on \(X\) by \(x \preceq y\) iff \(y \leq x\), the function \(f : X \to X\) by

\[
f(x) = \ln(1 + \frac{x}{10})
\]

and the altering distance functions \(\psi, \varphi : [0, +\infty) \to [0, +\infty)\) by \(\psi(t) = \ln(1 + \frac{1}{2} \ln(1 + t))\) and \(\varphi(t) = \frac{t}{1000}\). Then, we have the following:

1. \((X, \preceq, d)\) is a partially ordered \(p\)-complete \(p\)-metric space.
2. \(f\) is an ordered increasing mapping.
3. \(f\) is continuous.
4. \(f\) is an ordered \((\psi, \varphi)\)-\(\Omega\)-contractive mapping, that is,

\[
\psi(\Omega(\ddot{d}(fx, fy))) \leq \psi(M(x, y)) - \varphi(M(x, y))
\]

for all \(x, y \in X\) with \(x \preceq y\), where

\[
M(x, y) = \max \left\{ \ddot{d}(x, y), \ddot{d}(x, fx), \ddot{d}(y, fy), \ddot{d}(y, fx) \right\}.
\]
**Proof.** The proof of (1), (2) and (3) is clear.

To prove (4), let \( x, y \in X \) with \( x \preceq y \). So, \( y \leq x \). Thus, using the mean value theorem for function \( \ln(1 + \frac{1}{10}) \), we have

\[
\psi\left( \Omega(\tilde{d}(fx, fy)) \right) = \ln \left( 1 + \frac{1}{2} \ln \left( 1 + e^{2e\left[ \ln \left( 1 + \frac{1}{10} \right) \right]} \left( 1 - \frac{1}{2} \ln \left( 1 + \frac{1}{10} \right) \right)^2 - 1 \right) \right)
\]

\[
= \ln \left( 1 + \frac{1}{2} \left[ \ln \left( 1 + e^{2e\left[ \ln \left( 1 + \frac{1}{10} \right) \right]} \left( 1 - \frac{1}{2} \ln \left( 1 + \frac{1}{10} \right) \right)^2 - 1 \right) \right] \right)
\]

\[
= \ln \left( 1 + \frac{1}{2} \left[ \ln(1 + \frac{1}{10}) - \ln(1 + \frac{1}{10}) \right]^2 \right)
\]

\[
\leq \frac{1}{100} |x - y|^2
\]

\[
\leq \frac{1}{100} \left[ e^{10} - 1 \right]
\]

\[
\leq \frac{1}{100} \left[ M(x, y) \right]
\]

\[
\leq \ln(1 + \frac{1}{2} \ln(1 + M(x, y))) - \frac{1}{1000} M(x, y)
\]

\[
= \psi(M(x, y)) - \varphi(M(x, y)).
\]

(34)

So, we conclude that \( f \) is a \((\psi, \varphi)\Omega\)-contractive mapping. Thus, all the hypotheses of Theorem 1 are satisfied and hence \( f \) has a fixed point. Indeed, 0 is the unique fixed point of \( f \).

**Remark 1** A subset \( W \) of a partially ordered set \( X \) is said to be well ordered if every two elements of \( W \) are comparable. Note that in Theorems 1 and 2, \( f \) has a unique fixed point provided that the fixed points of \( f \) are comparable.

**Example 5** Let \( X = \{0, 1, 2, 3\} \) be equipped with the following partial order \( \preceq \):

\[
\preceq := \{(0, 0), (1, 0), (1, 1), (1, 2), (2, 2), (3, 1), (3, 2), (3, 3)\}.
\]

Define the metric \( \tilde{d} : X \times X \to \mathbb{R}^+ \) by

\[
\tilde{d}(x, y) = \begin{cases} 
0, & x = y, \\
x + y, & x \neq y
\end{cases}
\]

(35)

and let \( \rho(x, y) = \sinh \tilde{d}(x, y) \). It is easy to see that \((X, \rho)\) is a \(p\)-complete \(p\)-metric space.

Define the self-map \( f \) by

\[
f = \begin{pmatrix} 
0 & 1 & 2 & 3 \\
0 & 0 & 1 & 1
\end{pmatrix}
\]

We see that \( f \) is an ordered increasing mapping and \((X, \preceq, p)\) enjoys the s.l.c.p.. Define \( \psi, \varphi : [0, \infty) \to [0, \infty) \) by \( \psi(t) = \sqrt{t} \) and \( \varphi(t) = \frac{1}{1+t^2} \). One can easily check that \( f \) is a \((\psi, \varphi)\Omega\)-contractive mapping. Indeed, we have some cases as follows:
1. \((x, y) = (3, 1)\). Then,

\[
\psi(\Omega(\rho(fx, fy))) = \sqrt{\sinh(f3 + f1)}
\]
\[
= \sqrt{\sinh(1 + 0)}
\]
\[
= 1.08406696917
\]
\[
\leq 5.22397522938 - 0.00134095068
\]
\[
= \sqrt{M(x, y)} - \frac{1}{1 + (M(x, y))^2}
\]
\[
= \psi(M(x, y)) - \varphi(M(x, y)).
\]

2. \((x, y) = (3, 2)\). Then,

\[
\psi(\Omega(d(fx, fy))) = \sqrt{\sinh(f3 + f2)}
\]
\[
= \sqrt{\sinh(1 + 1)}
\]
\[
= 1.90443178083
\]
\[
\leq 8.6141285443 - 0.00018158323
\]
\[
= \sqrt{M(x, y)} - \frac{1}{1 + (M(x, y))^2}
\]
\[
= \psi(M(x, y)) - \varphi(M(x, y)).
\]

Thus, all the conditions of Theorem 2 are satisfied and hence \(f\) has a fixed point. Indeed, 0 is the fixed point of \(f\).

3 Existence theorem for a solution of an integral equation

Consider the integral equation

\[
x(t) = p(t) + \int_0^T \lambda(t, r)f(r, x(r))dr, \quad t \in [0, T]
\]

where \(0 < T\). The purpose of this section is to give an existence theorem for a solution of 38 that belongs to \(X = C(I, \mathbb{R})\) (the set of continuous real functions defined on \(I = [0, T]\)), via the obtained result in Theorem 2. Obviously, this space with the \(p\)-metric given by

\[
\rho(x, y) = e^\left(\max_{t \in I} \left|x(t) - y(t)\right|\right) - 1
\]

for all \(x, y \in X\) is a \(p\)-complete \(p\)-metric space with \(\Omega(t) = e^t - 1\).

We endow \(X\) with the partial order \(\preceq\) given by

\[
x \preceq y \iff x(t) \leq y(t),
\]

for all \(t \in I\). \((X, \preceq, \rho)\) is regular [25]. We will consider 38 under the following assumptions:

(i) \(f, p : [0, T] \times \mathbb{R} \to \mathbb{R}\) are continuous.

(ii) \(\lambda : [0, T] \times \mathbb{R} \to [0, \infty)\) is continuous.
There exists \( k \in (0, 1) \) such that for all \( x, y \) with \( x \preceq y \)
\[
0 \leq e^{\int_0^T \lambda(t,r) [f(r,x(r))-f(r,y(r))]dr} - 1 \leq k [e^{\varphi(y(t))} - x(t)] - 1,
\]
and \( \ln(1 + t) - 2kt \geq 0 \) for all \( t \in I \).

There exists continuous function \( \alpha : [0, T] \to \mathbb{R} \) such that
\[
\alpha(t) \leq \varphi(t) + \int_0^T \lambda(t,r) f(r, \alpha(r))dr.
\]

\[\text{Theorem 3}\]
Under assumptions (i)-(v), \( 38 \) has a solution in \( X \), where \( X = C([0, T], \mathbb{R}) \).

\[\text{Proof.}\]
We define \( F : X \to X \) by
\[
F(x(t)) = p(t) + \int_0^T \lambda(t,r) f(r, x(r))dr.
\]
The mapping \( F \) is ordered increasing since, for \( x \preceq y \)
\[
f(t, x) \leq f(t, y),
\]
and from \( \lambda(t, r) > 0 \), we have
\[
F(x(t)) = p(t) + \int_0^T \lambda(t,r) f(r, x(r))dr \leq p(t) + \int_0^T \lambda(t,r) f(r, y(r))dr = F(y(t)).
\]

Now, we have
\[
\psi\left(\Omega(\rho(Fx(t), Fy(t)))\right) = \ln \left( \Omega(e^{[\rho(Fx(t), Fy(t))] - 1} + 1 \right)
\leq e^{\left[\int_0^T \lambda(t,r) [f(r,x(r))-f(r,y(r))]dr\right] - 1}
\leq k [e^{\varphi(y(t))} - x(t)] - 1
\leq k \rho(x, y)
\leq kM(x, y)
\leq \ln(M(x, y) + 1) - kM(x, y)
= \psi(M(x, y)) - \varphi(M(x, y)).
\]

where
\[
M(x, y) = \max \left\{ \rho(x, y), \rho(x, Fx), \rho(y, Fy), \rho(y, Fx) \right\}.
\]

Let \( \alpha \) be the function appearing in assumption (v). Then we get
\[
\alpha \preceq F(\alpha).
\]

Thus, from Theorem 2 by \( \psi(t) = \ln(1 + t) \) and \( \varphi(t) = kt \) we deduce the existence of an \( x \in X \) such that \( x = F(x) \).
References