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A Note on Essentially Left ϕ-Contractible Banach Algebras

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Abstract In this note, we show that [11, Corollary 3.2] is not always true. In fact, we characterize essential left ϕ-contractibility of group algebras in terms of compactness of its related locally compact group. Also, we show that for any compact commutative group G, $L^2(G)$ is always essentially left ϕ-contractible. We discuss the essential left ϕ-contractibility of some Fourier algebras.

Keywords Group algebra · Essential left ϕ-contractible · Banach algebra

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1 Introduction and preliminaries

Johnson introduced and studied the notion of amenability for Banach algebras. A Banach algebra A is called amenable, if every continuous (bounded linear) derivation D from A into X^* is inner, that is, D has a form

$$D(a) = a \cdot x_0 - x_0 \cdot a \quad (a \in A),$$

for some $x_0 \in X^*$, where X is a Banach A-bimodule. For the history of amenability of Banach algebras, see [10].

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Ghahramani and Loy in [2] defined a generalized notion of amenability for Banach algebras called essential amenability, that is, every continuous derivation D from A into X^* is inner, where X is an arbitrary neo-unital Banach A-bimodule ($X = A \cdot X \cdot A$).

Kanuith et. al. in [5] defined and investigated the notion of left ϕ-amenability for a Banach algebra A, where ϕ is a non-zero multiplicative linear functional. Indeed a Banach algebra A is left ϕ-amenable if every derivation $D : A \to X^*$ is inner, where X is a Banach A-bimodule with the left module action $a \cdot x = \phi(a)x$ for all $a \in A, x \in X$. It is known that for a locally compact group G, the group algebra $L^1(G)$ is left ϕ-amenable if and only if G is amenable. Also the Fourier algebra $A(G)$ is always left ϕ-amenable, see [5] and [12].

Motivated by these considerations Nasr-isfahani et. al. in [6] introduced the concept of essential left ϕ-amenability for Banach algebras. A Banach algebra A is called essentially left ϕ-amenable if every derivation $D : A \to X^*$ is inner, where X is a neo-unital Banach A-bimodule with the left module action $a \cdot x = \phi(a)x$ for all $a \in A, x \in X$. Nasr-isfahani et. al. studied some Banach algebras related to a locally compact groups under the concept of essential left ϕ-amenability.

Recently R. Sadeghi Nahrkhalaji defined the concept of essential left ϕ-contractible for Banach algebras. A Banach algebra A is called essentially left ϕ-contractible if every continuous derivation $D : A \to X$ is inner, where X is a neo-unital Banach A-bimodule with the right module action $x \cdot a = \phi(a)x$ for all $a \in A, x \in X$, see [11]. R. Sadeghi Nahrkhalaji studied the essentially left ϕ-contractibility of some Banach algebras related to a locally compact group. Also some hereditary properties of this new notion are given in [11].

In this paper, we study essentially left ϕ-contractibility of Banach algebras. We show that [11, Corollary 3.2] is not always true. In fact, we characterize essential left ϕ-contractibility of the the group algebras in terms of compactness of its related locally compact group. Also we show that for any compact commutative group G, $L^2(G)$ is always essentially left ϕ-contractible. We discuss essential left ϕ-contractibility of some Fourier algebras.

We give some notations and definitions that we use in this paper frequently. Suppose that A is a Banach algebra. Throughout this manuscript, the character space of A is denoted by $\Delta(A)$, that is, all non-zero multiplicative linear functionals (characters) on A.

The projective tensor product $A \otimes_p A$ is a Banach A-bimodule via the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A).$$

The product morphism $\pi_A : A \otimes_p A \to A$ is given by $\pi_A(a \otimes b) = ab$, for every $a, b \in A$. Let X and Y be Banach $A-$bimodules. The map $T : X \to Y$ is called $A-$bimodule morphism, if

$$T(a \cdot x) = a \cdot T(x), \quad T(x \cdot a) = T(x) \cdot a, \qquad (a \in A, x \in X).$$

2 Essential left ϕ-contractibility

Note that the Cohen-Hewit factorization is valid, whenever the Banach algebra A has a "bounded" left approximate identity, see [4, Theorem 1.1.4, p2]. Then in [11, Proposition 2.3] to show that $A \otimes_p A$ is neo-unital, A must have a bounded approximate identity. So we state the correct version of [11, Proposition 2.3] here.

Theorem 1 *Let A be a Banach algebra with a bounded approximate identity and $\phi \in \Delta(A)$. Then A is left ϕ-contractible if and only if A is essentially left ϕ-contractible.*

Proof. See the proof of [11, Proposition 2.3]. □

Let G be a locally compact group and $L^1(G)$ be its associated group algebra. We denote $\widehat{G}$ for the dual group of G, that is, the set of all non-zero continuous homomorphisms ρ from G into $T = \{z \in C : |z| = 1\}$. It is known that every non-zero multiplicative linear functional on $L^1(G)$ has the form ϕ_ρ for some $\rho \in \widehat{G}$, where

$$\phi_\rho(f) = \int_G \overline{\rho(x)} f(x) dx, \quad f \in L^1(G),$$

where dx is denoted for the Haar measure. For more information about the characters of group algebra see [3, Theorem 23.7].

We should remind that a Banach algebra is left ϕ-contractible if and only if there exists an element $m \in A$ such that $am = \phi(a)m$ and $\phi(m) = 1$ for all $a \in A$. For knowing more about left ϕ-contractibility of a Banach algebra and its hereditary properties through the homological approach, see [7].

Theorem 2 *Let G be a locally compact group. Then $L^1(G)$ is essentially ϕ-contractible if and only if G is compact.*

Proof. Let G be a compact group. Then each continuous homomorphism $\rho : G \to T$ belongs to $L^\infty(G)$. On the other hand $L^\infty(G) \subseteq L^1(G)$. So $\rho \in L^1(G)$. Consider

$$f * \rho(x) = \int_G f(y)\rho(y^{-1}x)dy = \int_G f(y)\rho(y^{-1})\rho(x)dy$$

and

$$\int_G f(y)\rho(y^{-1})\rho(x)dy = \rho(x)\int_G f(y)\overline{\rho(y)}dy = \rho(x)\phi_\rho(f).$$

It follows that $f * \rho = \phi_\rho(f)\rho$. Also

$$\phi_\rho(\rho) = \int_G \rho(x)\overline{\rho(x)}dy = \int_G \rho(x)\rho(x^{-1})dy = \int_G 1dx = 1,$$

here we consider the normalized Haar measure on G. Thus by [7, Theorem 2.1] $L^1(G)$ is left ϕ_ρ-contractible. So $L^1(G)$ is essentially left ϕ-contractible

Conversely, suppose that $L^1(G)$ is essentially left ϕ_ρ-contractible. Since $L^1(G)$ has a bounded approximate identity, by [11, Proposition 2.3], essential left ϕ_ρ-contractibility of $L^1(G)$ implies the left ϕ_ρ-contractibility of $L^1(G)$. Applying [1, Theorem 3.3] follows that G is compact.

□

In the following theorem, we show that $ii \Rightarrow iv$ of [11, Corollary 3.2] is not valid just only for a finite group G. Suppose that G is a locally compact group. It is well-known that $L^2(G)$ is a Banach algebra with convolution if and only if G is compact.

Theorem 3 *Let G be a compact commutative group. Then $L^2(G)$ is essentially left ϕ_ρ-contractible, for each $\phi_\rho \in \Delta(L^2(G))$.*

Proof. Let $L^2(G)$ be essentially left ϕ_ρ-contractible. It is known by Plancherel Theorem [9, Theorem 1.6.1] that $L^2(G)$ is isometrically isomorphic to $\ell^2(\widehat{G})$, where $\ell^2(\widehat{G})$ is equipped with the pointwise multiplication. Zhang in [13, Example] showed that $\ell^2(\widehat{G})$ is approximately biprojective, that is, there exists a net $\rho_\alpha : \ell^2(\widehat{G}) \to \ell^2(\widehat{G}) \otimes_p \ell^2(\widehat{G})$ of $\ell^2(\widehat{G})$-bimodule morphisms such that $\pi_{\ell^2(\widehat{G})} \circ \rho_\alpha(a) \to a$, for each $a \in \ell^2(\widehat{G})$, where $\pi_{\ell^2(\widehat{G})} : \ell^2(\widehat{G}) \otimes_p \ell^2(\widehat{G}) \to \ell^2(\widehat{G})$ is the product morphism given by $\pi_{\ell^2(\widehat{G})}(a \otimes b) = ab$ for all $a, b \in \ell^2(\widehat{G})$. On the other hand suppose that Λ is the collection of all finite subsets of $\widehat{G}$. Clearly with the inclusion Λ becomes an ordered set. One can see that

$$\{u_\beta = \sum_{i \in \beta} e_i : \beta \in \Lambda\},$$

here e_i is an element of $\ell^2(G)$ equal to 1 at i and 0 elsewhere, forms a central approximate identity for $\ell^2(\widehat{G})$. Then for each $\phi_\rho \in \Delta(L^2(G))$ we can find an element $x_0 \in L^2(G)$ such that $ax_0 = x_0 a$ and $\phi_\rho(x_0) = 1$ for each $a \in L^2(G)$. Applying [8, Lemma 3.5] follows that $L^2(G)$ is ϕ_ρ-contractible, for each $\phi_\rho \in \Delta(L^2(G))$. Therefore $L^2(G)$ is essentially left ϕ_ρ-contractible, for each $\phi_\rho \in \Delta(L^2(G))$. □

Theorem 4 *Let G be an amenable group and $A(G)$ be the Fourier algebra on G. Then $A(G)$ is essentially left ϕ-contractible if and only if G is discrete.*

Proof. Let $A(G)$ be essentially left ϕ-contractible. Since G is amenable by Leptin's Theorem ([10, Theorem 7.1.3]), amenability of G implies that $A(G)$ has a bounded approximate identity. By [11, Proposition 2.3] essentially left ϕ-contractibility of $A(G)$ gives that $A(G)$ is left ϕ-contractible. Using [1, Theorem 3.5] shows that G is discrete.

Converse is clear by [1, Theorem 3.5]. □

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