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Coupled fixed point theorems on G-metric spaces via α-series

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Abstract The object of this paper is to study of the results of coupled fixed point in generalized metric spaces, as known as G-metric spaces. We will impose some conditions upon a self-mapping and a sequence of mappings via a kind of series, known as a-series. Also, an example is provided to illustrate the results.

Keywords G-metric space · α-series · Couple fixed point · Couple coincidence point · compatible mappings

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1 Introduction

The concept of G-metric spaces, as a generalization of the metric space (X, d), was introduced in [8] and [9]. In [2], Bhaskar and Lakshmikantham introduced coupled fixed point results for partially ordered metric spaces. Latter, many authors have acquired interesting important coupled fixed point theorems [4]-[6]

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In this paper, we investigate coupled fixed point theorems by imposing some conditions on a self-mapping g and a sequence of mappings $\{T_m\}_{m\in\mathbf{N}_0}$, for partially ordered G-metric spaces, via α-series. The α-series are wider than the convergent series. Throughout this article, $\mathbf{N}$ is a positive integer and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, and "$\xrightarrow{G}$" will denote G-convergence. First, we present some basic definitions that are used throughout the paper.

Definition 1 [9] A G-metric space is a pair (X, G), where $X \neq \emptyset$ and $G : X^3 \to [0, +\infty)$ is map such that satisfies:

(G1) $G(x^1, x^2, x^3) = 0$ if $x^1 = x^2 = x^3$.

(G2) $G(x^1, x^1, x^2) > 0$ for all $x^1, x^2 \in X$ with $x^1 \neq x^2$.

(G3) $G(x^1, x^1, x^2) \leq G(x^1, x^2, x^3)$ for all $x^1, x^2, x^3 \in X$ with $x^3 \neq x^2$.

(G4) $G(x^1, x^2, x^3) = G(x^1, x^3, x^2) = G(x^2, x^3, x^1) = \dots$ (symmetry in all three variables).

(G5) $G(x^1, x^2, x^3) \leq G(x^1, e, e) + G(e, x^2, x^3)$ for all $x^1, x^2, x^3, e \in X$, (rectangle inequality).

The map G is called a G-metric on X.

Proposition 1 *[9] Every G-metric space (X, G) defines a metric space (X, d_G) by $d_G(x^1, x^2) = G(x^1, x^2, x^2) + G(x^2, x^1, x^1)$, for all $x^1, x^2 \in X$.*

Definition 2 [9] Let (X, G) be a G-metric space. Then

1. a sequence $\{x^1_m\} \in X$ is G-convergent to x^1 if
$$\lim_{m,n\to+\infty} G(x^1, x^1_m, x^1_m) = 0,$$
that is, for each $\epsilon > 0$ there exists N such that $G(x^1, x^1_m, x^1_m) < \epsilon$ for all $m, n \geq N$.
2. If for each $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x^1_m, x^1_m, x^1_m) < \epsilon$ for all $m, n \geq N$, then a sequence $\{x^1_m\}$ is called G-Cauchy.
3. If every G-Cauchy sequence in (X, G) be G-convergent in (X, G), then G-metric space is called G-complete.

Proposition 2 *[9] A G-metric space (X, G) is G-complete if and only if (X, d_G) is a complete metric space.*

Definition 3 [9] Let (X, G) be a G-metric space, and let a mapping $F : X^2 \to X$. F is called continuous if sequences $x^1_n \xrightarrow{G} x^1, x^2_n \xrightarrow{G} x^2$, then sequence $F(x^1_m, x^2_m) \xrightarrow{G} F(x^1, x^2)$.

Definition 4 [9] Let (X, G) and (X', G') be G-metric spaces, and let a function $f : (X, G) \to (X', G')$. Then f is called G-continuous at a point $e \in X$ iff for every $\epsilon > 0$, there exists $\delta > 0$ such that $x^1, x^2 \in X$ and $G(e, x^1, x^2) < \delta$ implies $G(f(e), f(x^1), f(x^2)) < \epsilon$.
A function f is G-continuous at X iff it is G-continuous at all $e \in X$.

Proposition 3 *[9] Let (X, G) be a G-metric space. Then the following are equivalent:*

1. *sequence* $x_n^1 \xrightarrow{G} x^1$*;*
2. $G(x_m^1, x_m^1, x^1) \to 0$ *as* $n \to +\infty$*;*
3. $G(x_m^1, x^1, x^1) \to 0$ *as* $n \to +\infty$*;*
4. $G(x_m^1, x_m^1, x^1) \to 0$ *as* $n, m \to +\infty$.

Definition 5 [3] let $F : X^2 \longrightarrow X$. An element $(x^1, x^2) \in X^2$ is called a *coupled fixed point* of F if

$$F(x^1, x^2) = x^1, \quad F(x^2, x^1) = x^2.$$

Definition 6 [7] Let $F : X^2 \longrightarrow X$ and $g : X \longrightarrow X$ are given. An element $(x^1, x^2) \in X^2$ is called a *coupled coincidence point* of the mappings F and g if $F(x^1, x^2) = gx^1$ and $F(x^2, x^1) = gx^2$. So, (gx^1, gx^2) is called a *coupled coincidence point.*

Definition 7 Let $(X, \preceq)$ be a poset (or partially ordered set) and $F : X^2 \to X$. We say that F has the *mixed monotone property* if for any $x^1, x^2 \in X$

$$\begin{aligned} x_1^1, x_2^1 \in X, \quad x_1^1 \preceq x_2^1 \quad &\Rightarrow \quad F(x_1^1, x^2) \preceq F(x_2^1, x^2), \\ x_1^2, x_2^2 \in X, \quad x_1^2 \preceq x_2^2 \quad &\Rightarrow \quad F(x^1, x_1^2) \succeq F(x^1, x_2^2) \end{aligned}$$

that is, $F(x^1, x^2)$ is monotone increasing in x^1 and is monotone decreasing in x^2.

Definition 8 [7] Let $(X, \preceq)$ be a poset, $g : X \longrightarrow X$ and $F : X^2 \longrightarrow X$ are given. We say that F has the *g-mixed monotone property* if for any $x^1, x^2 \in X$,

$$\begin{aligned} x_1^1, x_2^1 \in X, \quad gx_1^1 \preceq gx_2^1 \quad &\Rightarrow \quad F(x_1^1, x^2) \preceq F(x_2^1, x^2), \\ x_1^2, x_2^2 \in X, \quad gx_1^2 \preceq gx_2^2 \quad &\Rightarrow \quad F(x^1, x_1^2) \succeq F(x^1, x_2^2) \end{aligned}$$

that is, $F(x^1, x^2)$ is monotone increasing in x^1, and it is monotone decreasing in x^2.

Definition 9 [3] Let (X, d) be a metric space and let $g : X \longrightarrow X$ and $F : X^2 \longrightarrow X$. The mappings F and g are said to be *compatible* if

$$\lim_{m \to +\infty} d(g(F(x_m^1, x_m^2)), F(gx_m^1, gx_m^2)) = 0,$$
$$\lim_{m \to +\infty} d(g(F(x_m^2, x_m^1)), F(gx_m^2, gx_m^1)) = 0,$$

whenever $\{x_m^1\}, \{x_m^2\}$ are sequences in X, such that

$$\lim_{m \to +\infty} F(x_m^1, x_m^2) = \lim_{m \to +\infty} gx_m^1 = x^1,$$

$$\lim_{m\to+\infty} F(x_m^2, x_m^1) = \lim_{m\to+\infty} gx_m^2 = x^2,$$

for some $x^1, x^2 \in X$.

Definition 10 [3] Let $g : X \longrightarrow X$ and $F : X^2 \longrightarrow X$ be mappings, if

$$\lim_{m\to+\infty} g(F(x_m^1, x_m^2)) = gx^1, and \lim_{m\to+\infty} F(gx_m^1, gx_m^2) = F(x^1, x^2)$$

$$\lim_{m\to+\infty} g(F(x_m^2, x_m^1)) = gx^2, and \lim_{m\to+\infty} F(gx_m^2, gx_m^1) = F(x^2, x^1)$$

whenever $\{x_m^1\}, \{x_m^2\}$ are sequences in X, such that

$$\lim_{m\to+\infty} F(x_m^1, x_m^2) = \lim_{m\to+\infty} gx_m^1 = x^1,$$

$$\lim_{m\to+\infty} F(x_m^2, x_m^1) = \lim_{m\to+\infty} gx_m^2 = x^2,$$

for some $x^1, x^2 \in X$, then g and F are called *reciprocally continuous*, and if

$$\lim_{m\to+\infty} g(F(x_m^1, x_m^2)) = gx^1, or \lim_{m\to+\infty} F(gx_m^1, gx_m^2) = F(x^1, x^2),$$

$$\lim_{m\to+\infty} g(F(x_m^2, x_m^1)) = gx^2, or \lim_{m\to+\infty} F(gx_m^2, gx_m^1) = F(x^2, x^1),$$

such that

$$\lim_{m\to+\infty} F(x_m^1, x_m^2) = \lim_{m\to+\infty} gx_m^1 = x^1,$$

$$\lim_{m\to+\infty} F(x_m^2, x_m^1) = \lim_{m\to+\infty} gx_m^2 = x^2,$$

for some $x^1, x^2 \in X$, then g and $F$are called *w-reciprocally continuous.*

Definition 11 [12] Let $(X, G, \preceq)$ be a partially ordered G-metric space on X. We say that $(X, G, \preceq)$ is regular if the following conditions hold:

1. if a increasing sequence $x_m^1 \to x^1$, then $x_m^1 \preceq x^1$ for all m,
2. if a decreasing sequence $x_m^2 \to x^2$, then $x_m^2 \succeq x^2$ for all m.

Definition 12 [11] Let $\{a_n\}$ be a sequence of positive real numbers. A series $\sum_{n=1}^{+\infty} a_n$ is called an α-series, if there exist $0 < \alpha < 1$ and $n_\alpha \in \mathbf{N}$ such that $\sum_{i=1}^{k} a_i \leq \alpha k$ for each $k \geq n_\alpha$.

For example, we know that every convergent series is bounded hence every convergent series is an α-series. Moreover, we can show that for each α, $0 < \alpha < 1$, $\sum_{n=1}^{+\infty} \frac{1}{n}$ is an α-series. In other words, there exsist divergent series that is an α-series.

In this paper, for the sequence of mappings $T_m : X^2 \to X$ and $g : X \to X$, where (X, G) is a G-metric space, we consider existence and uniqueness of coupled common fixed point.

2 Main results

Inspired by the Definition 8 we have the following definition.

Definition 13 Let $(X, \preceq)$ be a poset and $T_m : X^2 \to X$, $m \in \mathbf{N}_0$, and $g : X \to X$ are given. We say that T_m has the *g-mixed monotone property* if for any $x^1, x'^1, x^2, x'^2 \in X$,

$$gx^1 \preceq gx'^1,\ gx'^2 \preceq gx^2 \text{ imply } T_m(x^1, x^2) \preceq T_{m+1}(x'^1, x'^2),$$
$$T_{m+1}(x'^2, x'^1) \preceq T_m(x^2, x^1).$$

Definition 14 Let $T_m : X^2 \to X$ and $g : X \longrightarrow X$ are given. We call $\{T_m\}_{m\in\mathbf{N}_0}$ and g are satisfied in (K) property if there exists $0 \le \beta_{m,m'}, \gamma_{m,m'} < 1$ for $m, m' \in \mathbf{N}_0$, such that

$$G(T_m(x^1, x^2), T_{m'}(u^1, u^2), T_{m'}(u^3, x^3)) \le \beta_{m,m'}[G(gx^1, T_m(x^1, x^2), T_m(x^1, x^2)) + G(gu^1, T_{m'}(u^1, u^2), T_{m'}(u^3, x^3))] + \gamma_{m,m'}(G(gx^1, gu^1, gu^3)) \tag{1}$$

for $x^1, x^2, u^1, u^2 \in X$ with $gx^1 \preceq gu^1, gu^2 \preceq gx^2$ or $gx^1 \succeq gu^1, gu^2 \succeq gx^2$.

Definition 15 We call g and T_0 have a non-decreasing transcendence point in the first component and non-increasing transcendence point in the second component, which we call g and T_0 have *mixed coupled transcendence point*, if there exist $x_0^1, x_0^2 \in X$ such that

$$T_0(x_0^1, x_0^2) \succeq gx_0^1,\ T_0(x_0^2, x_0^1) \preceq gx_0^2. \tag{2}$$

We begin with the following statement, which, in proof of the main theorem, considers the sequences that are made in the following way. Let $x_0^1, x_0^2 \in X$, such that condition (2) holds, since $T_0(X^2) \subseteq g(X)$, we can define $x_1^1, x_1^2 \in X$ such that $gx_1^1 = T_0(x_0^1, x_0^2)$, $gx_1^2 = T_0(x_0^2, x_0^1)$. Again since $T_0(X^2) \subseteq g(X)$, there exists $x_2^1, x_2^2 \in X$ such that $gx_2^1 = T_1(x_1^1, x_1^2)$, $gx_2^2 = T_1(x_1^2, x_1^1)$. Continuing this technique, for all $n \ge 0$, we get

$$gx_m^1 = T_{m-1}(x_{m-1}^1, x_{m-1}^2),\quad gx_m^2 = T_{m-1}(x_{m-1}^2, x_{m-1}^1). \tag{3}$$

Now, using mathematical induction, we show that

$$gx_m^1 \preceq gx_{m+1}^1,\quad gx_m^2 \succeq gx_{m+1}^2, \tag{4}$$

for all $n \ge 0$. To show this, since (2) holds in view of

$$gx_1^1 = T_0(x_0^1, x_0^2),\quad gx_1^2 = T_0(x_0^2, x_0^1),$$

we have $gx_0^1 \preceq gx_1^1$, $gx_0^2 \succeq gx_1^2$, that is, for $n = 0$, condition (4) holds. We assume that (4) holds for some $n > 0$. Now, by (3) and (4), we deduce that

$$gx_{m+1}^1 = T_m(x_m^1, x_m^2) \preceq T_{m+1}(x_{m+1}^1, x_{m+1}^2) = gx_{m+2}^1,$$
$$gx_{m+2}^2 = T_{m+1}(x_{m+1}^2, x_{m+1}^1) \preceq T_m(x_m^2, x_m^1) = gx_{m+1}^2.$$

Thus by mathematical induction, we have done. Therefore, we have

$$gx_0^1 \preceq \ldots \preceq gx_{m+1}^1 \preceq \ldots,$$

$$gx_0^2 \succeq \ldots \succeq gx_{m+1}^2 \succeq \ldots.$$

Due to the above considerations and [10] Definitions 9 and 10 are as follows.

Definition 16 Let (X, G) be a G-metric space, and let $g : X \to X$ and $T_m : X^2 \to X$ be given. g and $\{T_m\}_{m\in\mathbf{N}_0}$ are *compatible* if

$$\lim_{m\to+\infty} G(gT_m(x_m^1, x_m^2), T_m(gx_m^1, gx_m^2), T_m(gx_m^1, gx_m^2)) = 0,$$

$$\lim_{m\to+\infty} G(gT_m(x_m^2, x_m^1), T_m(gx_m^2, gx_m^1), T_m(gx_m^2, gx_m^1)) = 0,$$

whenever $\{x_m^1\}, \{x_m^2\}$ are sequences in X, such that

$$\lim_{m\to+\infty} T_m(x_m^1, x_m^2) = \lim_{m\to+\infty} gx_{m+1}^1 = gx^1,$$

$$\lim_{m\to+\infty} T_m(x_m^2, x_m^1) = \lim_{m\to+\infty} gx_{m+1}^2 = gx^2,$$

for some $x^1, x^2 \in X$.

Definition 17 Let (X, G) be a G-metric space, and let $g : X \to X$ and $T_m : X^2 \to X$ be given. g and $\{T_m\}_{m\in\mathbf{N}_0}$ are called *w-reciprocally continuous* if

$$\lim_{m\to+\infty} g(T_m(x_m^1, x_m^2)) = g(x^1),$$

$$\lim_{m\to+\infty} g(T_m(x_m^2, x_m^1)) = g(x^2),$$

whenever $\{x_m^1\}, \{x_m^2\}$ are sequences in X, such that

$$\lim_{m\to+\infty} T_m(x_m^1, x_m^2) = \lim_{m\to+\infty} g(x_{m+1}^1) = x^1,$$

$$\lim_{m\to+\infty} T_m(x_m^2, x_m^1) = \lim_{m\to+\infty} g(x_{m+1}^2) = x^2,$$

for some $x^1, x^2 \in X$.

Definition 18 For $x^1, x^2 \in X$, we say that (x^1, x^2) is coupled comparable with (u^1, u^2) iff

$$x^1 \succeq u^1, x^2 \preceq u^2 \ \textit{or}\ x^1 \preceq u^1, x^2 \succeq u^2 \ \textit{or}$$

$$x^1 \succeq u^2, x^2 \preceq u^1 \ \textit{or}\ x^1 \preceq u^2, x^2 \succeq u^1.$$

If in the above definition replace (x^1, x^2) and (u^1, u^2) with (gx^1, gx^2) and (gu^1, gu^2), we call (x^1, x^2) is coupled comparable with (u^1, u^2) with respect to g.

At the following, we state the main result of this manuscript.

Theorem 1 *Let $(X, G, \preceq)$ be partially ordered G-metric on X such that (X, G) is a complete G-metric space, Let $g : X \to X$ and $T_m : X^2 \to X$ be a sequence of mappings, has g-mixed monotone property with $T_m(X^2) \subseteq g(X)$, g and $\{T_m\}_{m\in\mathbf{N}_0}$ are continuous, w-reciprocally continuous, and $g(X)$ is closed. Assume that the following holds:*

1. *$\{T_m\}_{m\in\mathbf{N}_0}$ and g are compatible,*
2. *there exists $(x_0^1, x_0^2) \in X$ such that condition (2) holds,*
3. *$\{T_m\}_{m\in\mathbf{N}_0}$ and g satisfying the condition (K), if $\sum_{m=1}^{+\infty}\left(\frac{\beta_{m,m+1}+\gamma_{m,m+1}}{1-\beta_{m,m+1}}\right)$ is an α-series, then $\{T_m\}_{m\in\mathbf{N}_0}$ and g have a coupled coincidence point.*
4. *If $\{T_m\}_{m\in\mathbf{N}_0}$ and g, have coupled coincidence points comparable with respect to g, then there exists $(x^1, x^2) \in X$ such that*

$$x^1 = gx^1 = T_m(x^1, x^2), \quad x^2 = gx^2 = T_m(x^2, x^1),$$

for $m \in \mathbf{N}$, that is, $\{T_m\}_{m\in\mathbf{N}_0}$ and g have a unique coupled common fixed point.

Proof For any $x_0^1, x_0^2 \in X$, we can consider the sequences $\{x_r^1\}, \{x_r^2\}$ constructed above, that is,

$$gx_r^1 = T_{r-1}(x_{r-1}^1, x_{r-1}^2), \quad gx_r^2 = T_r(x_{r-1}^2, x_{r-1}^1).$$

By (1), we get

$$\begin{aligned} G(gx_1^1, gx_2^1, gx_2^1) &= G(T_0(x_0^1, x_0^2), T_1(x_1^1, x_1^2), T_1(x_1^1, x_1^2)) \\ &\le \beta_{1,2}[G(gx_0^1, T_0(x_0^1, x_0^2), T_0(x_0^1, x_0^2)) \\ &+G(gx_1^1, T_1(x_1^1, x_1^2), T_1(x_1^1, x_1^2))] + \gamma_{1,2}G(gx_0^1, gx_1^1, gx_1^1) \\ &= \beta_{1,2}[G(gx_0^1, gx_1^1, gx_1^1) + G(gx_1^1, gx_2^1, gx_2^1)] \\ &+\gamma_{1,2}G(gx_0^1, gx_1^1, gx_1^1). \end{aligned}$$

It follows that

$$G(gx_1^1, gx_2^1, gx_2^1) \le \left(\frac{\beta_{1,2} + \gamma_{1,2}}{1 - \beta_{1,2}}\right) G(gx_0^1, gx_1^1, gx_1^1). \tag{5}$$

Also, we get

$$\begin{aligned} G(gx_2^1, gx_3^1, gx_3^1) &= G(T_1(x_1^1, x_1^2), T_2(x_2^1, x_2^2), T_2(x_2^1, x_2^2)) \\ &\le (\frac{\beta_{2,3} + \gamma_{2,3}}{1 - \beta_{2,3}})G(gx_1^1, gx_2^1, gx_2^1) \\ &\le (\frac{\beta_{2,3} + \gamma_{2,3}}{1 - \beta_{2,3}})(\frac{\beta_{1,2} + \gamma_{1,2}}{1 - \beta_{1,2}})G(gx_0^1, gx_1^1, gx_1^1). \end{aligned}$$

Similarly, we obtain

$$G(gx_r^1, gx_{r+1}^1, gx_{r+1}^1) \le \prod_{m=1}^{r} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1-\beta_{m,m+1}} \right) G(gx_0^1, gx_1^1, gx_1^1). \quad (6)$$

Using the same method as above, we can also show that

$$G(gx_r^2, gx_{r+1}^2, gx_{r+1}^2) \le \prod_{m=1}^{r} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1-\beta_{m,m+1}} \right) G(gx_0^2, gx_1^2, gx_1^2). \quad (7)$$

Adding (6) and (7), we get

$$\begin{aligned} \delta_r &:= G(gx_r^1, gx_{r+1}^1, gx_{r+1}^1) + G(gx_r^2, gx_{r+1}^2, gx_{r+1}^2) \\ &= \sum_{i=1}^{2} G(gx_r^i, gx_{r+1}^i, gx_{r+1}^i) \\ &\le \prod_{m=1}^{r} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1-\beta_{m,m+1}} \right) \sum_{i=1}^{2} G(gx_0^i, gx_1^i, gx_1^i) \\ &= \prod_{m=1}^{r} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1-\beta_{m,m+1}} \right) \delta_0. \end{aligned}$$

Moreover, by repeated use of (G5) and for $p > 0$, we have

$$\begin{aligned} \sum_{i=1}^{2} G(gx_r^i, gx_{r+p}^i, gx_{r+p}^i) &\le \sum_{i=1}^{2} G(gx_r^i, gx_{r+1}^i, gx_{r+1}^i) \\ &\quad + \sum_{i=1}^{2} G(gx_{r+1}^i, gx_{r+2}^i, gx_{r+2}^i) \\ &\quad + \dots \\ &\quad + \sum_{i=1}^{2} G(gx_{r+p-1}^i, gx_{r+p}^i, gx_{r+p}^i) \\ &\le \prod_{m=1}^{r} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1-\beta_{m,m+1}} \right) \delta_0 \\ &\quad + \prod_{m=1}^{r+1} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1-\beta_{m,m+1}} \right) \delta_0 \\ &\quad + \dots \\ &\quad + \prod_{m=1}^{r+p-1} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1-\beta_{m,m+1}} \right) \delta_0 \end{aligned}$$

$$= \sum_{k=0}^{p-1} \prod_{m=1}^{r+k} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}} \right) \delta_0$$
$$= \sum_{k=r}^{r+p-1} \prod_{m=1}^{k} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}} \right) \delta_0.$$

Let α and n_α as in Definition 12, then, for $r \geq n_\alpha$ and using that the non-negative numbers geometric mean is less than or equal to the arithmetic mean, it follows that

$$\sum_{i=1}^{2} G(gx_r^i, gx_{r+p}^i, gx_{r+p}^i) \leq \sum_{k=r}^{r+p-1} \left[\frac{1}{k} \sum_{m=1}^{k} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}} \right) \right]^k \delta_0$$
$$\leq \left(\sum_{k=r}^{r+p-1} \alpha^k \right) \delta_0$$
$$\leq \frac{\alpha^r}{1-\alpha} \delta_0.$$

Now, letting $r \to +\infty$, we conclude that

$$\lim_{r\to\infty} \sum_{i=1}^{2} G(gx_r^i, gx_{r+p}^i, gx_{r+p}^i) = 0,$$

which implies that

$$\lim_{r\to\infty} G(gx_r^i, gx_{r+p}^i, gx_{r+p}^i) = 0,$$

Thus $\{gx_r^1\}, \{gx_r^2\}$ are Cauchy sequences in X. Since $g(X)$ is closed in a complete G-metric space, there exist $(x^1, x^2) \in X$, with $\lim_{r\to+\infty}\{gx_r^1\} = g(x^1) := x^1, \lim_{r\to+\infty}\{gx_r^2\} = g(x^2) := x^2$. By construction we have

$$\lim_{r\to\infty} g(x_{r+1}^1) = \lim_{r\to\infty} T_r(x_r^1, x_r^2) = x^1,$$
$$\lim_{r\to\infty} g(x_{r+1}^2) = \lim_{r\to\infty} T_r(x_r^2, x_r^1) = x^2.$$

Now, by w-reciprocally continuous and the compatibility of $\{T_m\}_{m\in\mathbf{N}_0}$ and g, we have

$$\lim_{r\to+\infty} T_r(g(x_r^1), g(x_r^2)) = g(x^1),$$
$$\lim_{r\to+\infty} T_r(g(x_r^2), g(x_r^1)) = g(x^2).$$

Now suppose that $\{T_m\}_{m\in\mathbf{N}}$ is continuous. Using triangle inequality we get

$$G(T_m(x^1, x^2), T_r(gx_r^1, gx_r^2), T_r(gx_r^1, gx_r^2)) \leq G[T_m(x^1, x^2), gT_r(x_r^1, x_r^2),$$
$$gT_r(x_r^1, x_r^2)] + G[gT_r(x_r^1, x_r^2), T_r(gx_r^1, gx_r^2), T_r(gx_r^1, gx_r^2)]$$

and

$$G(T_m(x^2, x^1), T_r(gx_r^2, gx_r^1), T_r(gx_r^2, gx_r^1)) \leq G[T_m(x^2, x^1), gT_r(x_r^2, x_r^1),$$

$$gT_r(x_r^2, x_r^1)] + G[gT_r(x_r^2, x_r^1), T_r(gx_r^2, gx_r^1), T_r(gx_r^2, gx_r^1)].$$

Now using continuity of $\{T_m\}_{m\in\mathbf{N}}$ and g, and taking the limit as $r \to \infty$ we get

$$G(T_m(x^1, x^2), gx^1, gx^1) = 0, \qquad G(T_m(x^2, x^1), gx^2, gx^2) = 0$$

i.e. $T_m(x^1, x^2) = gx^1$, and $T_m(x^2, x^1) = gx^2$. Thus, (x^1, x^2) is a coupled coincidence point of $\{T_m\}_{m\in\mathbf{N}}$ and g. Then the set of coupled coincidences is non-empty.

Now, we show that if (x^1, x^2) and (u^3, x^3) are coupled coincidence points, that is, if $gx^1 = T_m(x^1, x^2), gx^2 = T_m(x^2, x^1), gu^3 = T_m(u^3, x^3)$ and $gx^3 = T_m(x^3, u^3)$, then $gx^1 = gu^3$ and $gx^2 = gx^3$. Since the set of coupled coincidence points is comparable, applying condition (1), we get

$$G(gx^1, gu^3, gu^3) = G(T_m(x^1, x^2), T_{m'}(u^3, x^3), T_{m'}(u^3, x^3))$$

$$\leq \beta_{m,m'}[G(gx^1, T_m(x^1, x^2), T_m(x^1, x^2))$$

$$+G(gu^3, T_{m'}(u^3, x^3), T_{m'}(u^3, x^3))]$$

$$+\gamma_{m,m'} G(gx^1, gu^3, gu^3)$$

and so as $\gamma_{m,m'} < 1$, it follows that $G(gx^1, gu^3, gu^3) = 0$, that is, $gx^1 = gu^3$. Similarly, it can be proved that $gx^2 = gx^3$. Hence, $\{T_m\}_{m\in\mathbf{N}_0}$ and g have a unique coupled point of coincidence (gx^1, gx^1), since two compatible mappings, commute at their coincidence points. Thus, clearly, $\{T_m\}_{m\in\mathbf{N}_0}$ and g have a unique coupled common fixed point whenever $\{T_m\}_{m\in\mathbf{N}_0}$ and g are w-compatible.

As a result of Theorem 1, if g is the identity mapping, then we state the following corollary.

Corollary 1 *Let $(X, G, \preceq)$ be poset G-metric space on X such that (X, G) is a complete G-metric space. Let $\{T_m\}_{m\in\mathbf{N}_0}$ be a sequence of mappings from X^2 into X, which $\{T_m\}_{m\in\mathbf{N}_0}$ and $Id : X \to X$ satisfying the (K) property. Also, T_0 and Id have mixed coincidence point. If $\sum_{m=1}^{+\infty}\left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}}\right)$ is an α-series and X is regular, then $\{T_m\}_{m\in\mathbf{N}_0}$ has a coupled fixed point, that is, there exists $(x^1, x^2) \in X^2$ such that*

$$x^1 = T_m(x^1, x^2), x^2 = T_m(x^2, x^1), \text{ for } m \in \mathbf{N}_0.$$

Theorem 2 *Let $(X, G, \preceq)$ be partially ordered G-metric space on X such that $(X, G, \preceq)$ is regular and let g and $\{T_m\}_{m\in\mathbf{N}_0}$ be as in the preceding theorem and $lim_{r\to+\infty} \sup \beta_{r,m} < 1$. Therefore, the conditions $(1) - (3)$ in theorem 1 hold.*

Proof According to Theorem 1, sequences $\{gx_r^1\}$ and $\{gx_r^2\}$ are Cauchy sequences in the complete G-metric space $(g(X), G)$. Since $\{gx_r^1\}$ and $\{gx_r^2\}$ are non-decreasing and non-increasing respectively, using the regularity of $(X, G, \preceq)$, we have $gx_r^1 \preceq x^1, x^2 \preceq gx_r^2$ for all $r \geq 0$. Then by (1), we obtain

$$G(T_r(gx_r^1, gx_r^2), T_m(x^1, x^2), T_m(x^1, x^2))$$
$$\leq \beta_{r,m}[G(ggx_r, T_r(gx_r^1, gx_r^2), T_r(gx_r^1, gx_r^2))$$
$$+G(gx^1, T_m(x^1, x^2), T_m(x^1, x^2))] + \gamma_{r,m}(G(ggx_r, gx^1, gx^1)).$$

Taking the limit as $r \to +\infty$, we obtain $T_m(x^1, x^2) = gx^1$ as $\beta_{r,m} < 1$. Similarly, it can be proved that $gx^2 = T_m(x^2, x^1)$. Thus, (x^1, x^2) is a coupled coincidence point of $\{T_m\}_{m\in\mathbf{N}}$ and g.

Example 1 Let $X = [0, 1]$ and

$$G(x^1, x^2, x^3) = \max\{|x^1 - x^2|, |x^2 - x^3|, |x^3 - x^1|\}.$$

It is clear that (X, G) is a complete G-metric space. Also define $\beta_{m,m'} = \dfrac{1}{2^{2m+1}}, \gamma_{m,m'} = \frac{1}{2^m}$ for all $m, m' = 1, 2, \ldots$. Consider the mapping $T_m : X^2 \to X$ and $g : X \to X$ with

$$T_m(x^1, x^2) = \frac{x^1 + x^2}{2^m}, \ g(x^1) = 6x^1$$

for all $x^1, x^2 \in X$, $m = 1, 2, \ldots$.

$$G(T_m(x^1, x^2), T_{m'}(u^1, u^2), T_{m'}(u^3, x^3)) = |\frac{x^1 + x^2}{2^m} - \frac{u^3 + x^3}{2^{m'}}|$$

and

$$G(gx^1, T_m(x^1, x^2), T_m(x^1, x^2)) + G(gx^2, T_{m'}(u^1, u^2), T_{m'}(u^3, x^3))$$
$$= |6x^1 - \frac{x^1 + x^2}{2^m}| + |6x^2 - \frac{u^3 + x^3}{2^{m'}}|,$$
$$G(gx^1, gx^2, gu^3) = 6|x^1 - u^3|$$

Then by mathematical induction condition (1) is satisfied for all $x^1, x^2, u^1, u^2 \in X$ with $gx^1 \preceq gu^1, gu^2 \preceq gx^2$ or $gx^1 \succeq gu^1, gu^2 \succeq gx^2$ Moreover, the series

$$\sum_{m=1}^{+\infty} \left(\frac{\beta_{m,m+1} + \gamma_{m,m+1}}{1 - \beta_{m,m+1}} \right) = \sum_{m=1}^{+\infty} \frac{2^{m+1} + 1}{2^{2m+1} - 1}$$

is an α-series with $\alpha = \frac{1}{2}$. Then $(0, 0)$ is a unique coupled fixed point for T_m and g.

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