Global Analysis and Discrete Mathematics
Volume 6, Issue 1, pp. 13–23
ISSN: 2476-5341

On left ϕ-biflat and left ϕ-biprojectivity of θ-Lau product algebras

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Received: 9 October 2020 / Accepted: 29 November 2020

Abstract *Monfared* defined θ-Lau product structure $A \times_\theta B$ for two Banach algebras A and B, where $\theta : B \to C$ is a multiplicative linear functional. In this paper, we study the notion of left ϕ-biflatness and left ϕ-biprojectivity for the θ Lau product structure $A \times_\theta B$. For a locally compact group G, we show that $M(G) \times_\theta M(G)$ is left character biflat (left character biprojective) if and only if G is discrete and amenable (G is finite), respectively. Also we prove that $\ell^1(N_\vee) \times_\theta \ell^1(N_\vee)$ is neither $(\phi_{N_\vee}, \theta)$-biprojective nor $(0, \phi_{N_\vee})$-biprojective, where $\phi_{N_\vee}$ is the augmentation character on $\ell^1(N_\vee)$. Finally, we give an example among the Lau product structure of matrix algebras which is not left ϕ-biflat.

Keywords Left ϕ-amenability · Left ϕ-biflatnes · Left ϕ-biprojectivity · Left ϕ-contractibility · θ-Lau product.

Mathematics Subject Classification (2010) MSC Primary 43A07 · 46M10 · Secondary 43A20 · 46H05.

1 Introduction

Johnson defined amenable Banach algebras thorough virtual diagonals [8]. In fact a Banach algebra A is amenable, if there exists an element $M \in (A\hat{\otimes}A)^{**}$

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such that $a \cdot M = M \cdot a$ and $\pi_A^{**}(M)a = a$ for each $a \in A$, here π_A is given by $\pi_A(a \otimes b) = ab$ for each $a, b \in A$, see [14].

There are two homological notions parallel to amenability, namely biflatness and biprojectivity which were defined by Helemskii. In fact a Banach algebra A is called biflat (biprojective) if there exists a bounded A-bimodule morphism $\rho : A \to (A\hat{\otimes}A)^{**}(\rho : A \to A\hat{\otimes}A)$ such that $\pi_A^{**}\circ\rho(a) = a(\pi_A\circ\rho(a) = a)$, for all $a \in A$, respectively. It is well-known that a Banach algebra A is amenable if and only if A is biflat and A has a bounded approximate identity, see [14].

Recently some homological notions related to a multiplicative linear functional were given for Banach algebras. The notions like left ϕ-amenability, left ϕ-contractibility, left ϕ-biflatness and left ϕ-biprojectivity studied for the group algebras, the measure algebras and the Fourier algebras, for more information about these notions see [1], [7], [9], [13], [15] [16] and [17].

In this paper, we study the notion of left ϕ-biflatness and left ϕ-biprojectivity for the θ-Lau product structure $A\times_\theta B$. For a locally compact group G, we show that $M(G) \times_\theta M(G)$ is left character biflat (left character biprojective) if and only if G is discrete and amenable (G is finite), respectively. Also we prove that $\ell^1(N_\vee) \times_\theta \ell^1(N_\vee)$ is neither $(\phi_{N_\vee}, \theta)$-biprojective nor $(0, \phi_{N_\vee})$-biprojective, where ϕ_N is the augmentation character on $\ell^1(N)$. Finally, we give an example among the θ-Lau product structure of matrix algebras which is not left ϕ-biflat.

We remind some definitions and notations which we need in this paper. For an arbitrary Banach algebra A, the character space is denoted by $\sigma(A)$ consists of all non-zero multiplicative linear functionals on A and any element of $\sigma(A)$ is called a character. The $\theta-$Lau product was first introduced by Lau [10] for F-algebras. Monfared [12] introduced and investigated θ-Lau product space $A \times_\theta B$, for Banach algebras in general. Indeed for two Banach algebras A and B such that $\sigma(B) \neq \emptyset$ and θ be a non-zero character on B, the Cartesian product $A \times B$ by following multiplication and norm

$$(a, b)(a', b') = (aa' + \theta(b')a + \theta(b)a', bb'), \tag{1}$$

$$\|(a, b)\| = \|a\|_A + \|b\|_B \tag{2}$$

is a Banach algebra, for all $a, a' \in A$ and $b, b' \in B$. The Cartesian product $A\times B$ with the above properties called the $\theta-$Lau product of A and B which is denoted by $A\times_\theta B$. From [12] we identify $A\times\{0\}$ with A, and $\{0\}\times B$ with B. Thus, it is clear that A is a closed two-sided ideal while B is a closed subalgebra of $A\times_\theta B$, and $(A\times_\theta B)/A$ is isometrically isomorphic to B. If $\theta = 0$, then we obtain the usual direct product of A and B. Since direct products often exhibit different properties, we have excluded the possibility that $\theta = 0$. Moreover, if $B = C$, the complex numbers, and θ is the identity map on C, then $A \times_\theta B$ is the unitization $A^\sharp$ of A. Note that, by [12, Proposition 2.4], the character space $\sigma(A \times_\theta B)$ of $A \times_\theta B$ is equal to

$$\{(\phi, \theta) : \ \phi \in \sigma(A)\}\bigcup\{(0, \psi) : \ \psi \in \sigma(B)\}. \tag{3}$$

Also, the dual space $(A \times_\theta B)^*$ of $A \times_\theta B$ is identified with $A^* \times B^*$ such that for each $(a, b) \in A \times_\theta B$, $\phi \in \sigma(A)$ and $\psi \in \sigma(B)$ we have

$$\langle (\phi, \psi), (a, b) \rangle = \phi(a) + \psi(b). \tag{4}$$

Now, suppose that A^{**}, B^{**} and $(A \times_\theta B)^{**}$ are equipped with their first Arens products. Then $(A \times_\theta B)^{**}$ is isometrically isomorphic with $A^{**} \times_\theta B^{**}$. Also, for all $(m, n), (p, q) \in (A \times_\theta B)^{**}$ the first Arens product is defined by

$$(m, n)(p, q) = (mp + n(\theta)p + q(\theta)m, nq); \tag{5}$$

see [12, Proposition 2.12]. Note that every $\phi \in \sigma(A)$ has a unique extension to a character on A^{**} is given by $\tilde{\phi}$ where $\tilde{\phi}(m) = m(\phi)$, for all $m \in A^{**}$.

Note that A and B are closed two-sided ideal and closed subalgebra of $L := A \times_\theta B$, respectively. So, we can write $a = (a, 0)$ and $b = (0, b)$ for all $a \in A$ and $b \in B$. Therefore, $L = A \times_\theta B$ is a Banach $A-$bimodule and also is a Banach $B-$bimodule. It has worth to mention that some generalizations of twisted product related to a homomorphism are given recently but by [3] it seems those products are trivial.

We recall that if X is a Banach A-bimodule, then with the following actions X^* is also a Banach A-bimodule:

$$a \cdot f(x) = f(x \cdot a), \quad f \cdot a(x) = f(a \cdot x) \qquad (a \in A, x \in X, f \in X^*). \tag{6}$$

The projective tensor product of A with A is denoted by $A\widehat{\otimes}A$. The Banach algebra $A\widehat{\otimes}A$ is a Banach A-bimodule with the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A). \tag{7}$$

2 Left $\phi-$biflatness and left $\phi-$biprojectivity

In this note $p_A : L \longrightarrow A$ and $p_B : L \longrightarrow B$ are denoted for the usual projections given by $p_A(a, b) = a$ and $p_B(a, b) = b$. Suppose that $q_A : A \longrightarrow L$ and $q_B : B \longrightarrow L$ are injections defined by $q_A(a) = (a, 0)$ and $q_B(b) = (0, b)$. So q_A and p_B give

$$q_A \otimes q_A : A\widehat{\otimes}A \longrightarrow L\widehat{\otimes}L \tag{8}$$

and

$$p_B \otimes p_B : L\widehat{\otimes}L \longrightarrow B\widehat{\otimes}B \tag{9}$$

with

$$(q_A \otimes q_A)(a \otimes c) = (a, 0) \otimes (c, 0) \tag{10}$$

and

$$(p_B \otimes p_B)((a, b) \otimes (c, d)) = b \otimes d, \tag{11}$$

for all $a, c \in A$ and $b, d \in B$ respectively. It is easy to see that q_A and $q_A \otimes q_A$ are A-bimodule morphisms and p_B, q_B and $p_B \otimes p_B$ are B-bimodule morphisms.

The notion of left $\phi-$biprojectivity for Banach algebras first introduced by Sahami [17]. For a non-zero multiplicative linear functional ϕ on A, the Banach algebras A is called left $\phi-$biprojective if there exists a bounded linear map $\rho : A \longrightarrow A\widehat{\otimes}A$ such that

$$\rho(ab) = a \cdot \rho(b) = \phi(b)\rho(a), \quad \phi \circ \pi_A \circ \rho(a) = \phi(a), \quad (a, b \in A). \tag{12}$$

Proposition 1 *Let A and B be two Banach algebras which A has unit e. Also let $\phi \in \sigma(A)$ and $\theta \in \sigma(B)$. If L is left $(\phi, \theta)-$biprojective. Then A is left $\phi-$biprojective.*

Proof bounded linear map $\rho_L : L \longrightarrow L\widehat{\otimes}L$ such that $\rho_L(ab) = a \cdot \rho_L(b) = \phi(b)\rho_L(a)$ and $(\phi, \theta) \circ \pi_L \circ \rho_L = (\phi, \theta)$. We know that

$$r_{_A} \circ \pi_{_L} = \pi_A \circ (r_{_A} \otimes r_{_A}), \qquad \phi \circ r_{_A} = (\phi, \theta). \tag{13}$$

Define $\rho_A : A \longrightarrow A\widehat{\otimes}A$ by $\rho_A = (r_{_A} \otimes r_{_A}) \circ \rho_L \circ q_{_A}$. Consider

$$\begin{aligned} \rho_A(a_1a_2) &= (r_{_A} \otimes r_{_A}) \circ \rho_L \circ q_{_A}(a_1a_2) \\ &= (r_{_A} \otimes r_{_A}) \circ \rho_L(a_1 \cdot q_{_A}(a_2)) \\ &= a_1 \cdot (r_{_A} \otimes r_{_A}) \circ \rho_L(q_{_A}(a_2)) \\ &= a_1 \cdot \rho_A(a_2) \end{aligned}$$

and

$$\begin{aligned} \rho_A(a_1a_2) &= (r_{_A} \otimes r_{_A}) \circ \rho_L \circ q_{_A}(a_1a_2) \\ &= (r_{_A} \otimes r_{_A}) \circ \rho_L(q_{_A}(a_1) \cdot a_2) \\ &= \phi(a_2)(r_{_A} \otimes r_{_A}) \circ \rho_L(q_{_A}(a_1)) \\ &= \phi(a_2) \cdot \rho_A(a_1) \end{aligned}$$

for every a_1 and a_2 in A. So these facts follow that

$$\rho_A(a_1a_2) = a_1 \cdot \rho_A(a_2) = \phi(a_2)\rho_A(a_1). \tag{14}$$

Moreover we have

$$\begin{aligned} \phi \circ \pi_A \circ \rho_A(a) &= \phi \circ \pi_A \circ (r_{_A} \otimes r_{_A}) \circ \rho_L \circ q_{_A}(a) \\ &= (\phi \circ r_{_A} \circ \pi_L \circ \rho_L)(a, 0) \\ &= ((\phi, \theta) \circ \pi_L \circ \rho_L)(a, 0) \\ &= (\phi, \theta)(a, 0) \\ &= \phi(a), \end{aligned}$$

for all $a \in A$. Hence $\phi \circ \pi_A \circ \rho_A = \phi$. Therefore A is left ϕ-biprojective.

Proposition 2 *Let A and B be two Banach algebras $\psi \in \sigma(B)$. If L is left $(0, \psi)-$biprojective, then B is left $\psi-$biprojective. Converse holds whenever A is unital.*

Proof Suppose that L is left $(0,\psi)-$biprojective. Then there exists a bounded linear map $\rho_L : L \longrightarrow L\widehat{\otimes}L$ such that $(0,\psi)\circ\pi_L\circ\rho_L = (0,\psi)$. Define ρ_B : $B \longrightarrow B\widehat{\otimes}B$ by $\rho_B = (p_{_B}\otimes p_{_B})\circ\rho_L\circ q_{_B}$. Clearly

$$\pi_B\circ(p_{_B}\otimes p_{_B}) = p_{_B}\circ\pi_L, \qquad \psi\circ p_{_B} = (0,\psi). \tag{15}$$

Note that

$$\rho_B(b_1b_2) = b_1\cdot\rho_B(b_2) = \psi(b_2)\rho_B(b_1), \quad (b_1,b_2\in B). \tag{16}$$

Also $\psi\circ\pi_B\circ\lambda_B = \psi$. To see these facts, consider

$$\begin{aligned}\rho_B(b_1b_2) = (p_{_B}\otimes p_{_B})\circ\rho_L\circ q_{_B}(b_1b_2) &= (p_{_B}\otimes p_{_B})\circ\rho_L(q_{_B}(b_1)\cdot b_2)\\ &= \psi(b_2)(p_{_B}\otimes p_{_B})\circ\rho_L(q_{_B}(b_1)\\ &= \psi(b_2)\rho_B(a_1)\end{aligned}$$

and

$$\begin{aligned}\rho_B(b_1b_2) = (p_{_B}\otimes p_{_B})\circ\rho_L\circ q_{_B}(b_1b_2) &= (p_{_B}\otimes p_{_B})\circ\rho_L(b_1\cdot q_{_B}(b_2))\\ &= b_1\cdot(p_{_B}\otimes p_{_B})\circ\rho_L(q_{_B}(b_2),\\ &= b_1\cdot\rho_B(b_2),\end{aligned}$$

for all b_1 and b_2 in B. Moreover

$$\begin{aligned}\Big(\psi\circ\pi_B\circ\rho_B\Big)(b) &= \Big(\psi\circ\pi_B\circ(p_{_B}\otimes p_{_B})\rho_L\circ q_{_B}\Big)(b)\\ &= \Big(\psi\circ p_{_B}\circ\pi_L\circ\rho_L\Big)(0,b)\\ &= \Big((0,\psi)\circ\pi_L\circ\rho_L\Big)(0,b)\\ &= \psi(b),\end{aligned}$$

for all $b\in B$. For converse let B be left $\psi-$biprojective. Then there exists a bounded linear map $\rho_B : B \longrightarrow B\widehat{\otimes}B$ such that $\rho_B(ab) = a\cdot\rho_B(b) = \psi(b)\rho_B(a)$ and $\psi\circ\pi_B\circ\rho_B = \psi$. Define $\rho_L : L \longrightarrow L\widehat{\otimes}L$ via

$$\rho_L(a,b) := (S_{_B}\otimes S_{_B})\circ\rho_B(b),$$

for all $a\in A$ and $b\in B$. One can show that

$$\pi_L\circ(S_{_B}\otimes S_{_B}) = S_{_B}\circ\pi_B, \qquad (0,\psi)\circ S_{_B} = \psi, \quad ((S_{_B}\otimes S_{_B})\circ\rho_B(b))\cdot x = 0, \tag{17}$$

for all $b\in B$ and $x\in A$. Using these facts show that ρ_L is a bounded linear map such that

$$\rho_L(l_1l_2) = (0,\psi)(l_2)\rho_L(l_1) = l_1\cdot\rho_L(l_2), \tag{18}$$

for all $l_1,l_2\in L$. Also

$$(0,\psi)\circ\pi_L\circ\rho_L = (0,\psi). \tag{19}$$

It follows that L is left $(0,\psi)-$biprojective.

Remark 1 We claim that left $(\phi,\theta)-$biprojectivity of L gives that B is left $\theta-$biprojective. However it is easy but for the sake of completeness we give it here. We know that there exists a bounded linear map $\rho_L : L \longrightarrow L\widehat{\otimes}L$ such that

$$\rho_L(ab) = a\cdot\rho_L(b) = (\phi,\theta)(b)\rho_L(a), \quad (\phi,\theta)\circ\pi_L\circ\rho_L = (\phi,\theta), \qquad (a,b\in L). \tag{20}$$

On the other hand, one can see that

$$p_{_B}\circ\pi_L = \pi_B\circ(p_{_B}\otimes p_{_B}), \qquad r_A\circ\pi_L = \pi_A\circ(r_A\otimes r_A), \qquad \theta\circ p_{_B} = (0,\theta). \tag{21}$$

Let $\rho_B : B \longrightarrow B\widehat{\otimes}B$ be a map defined by $\rho_B := (p_{_B}\otimes p_{_B})\circ\rho_L\circ q_{_B}$. The fact $\Big((\phi,0)\circ\pi_L\circ\rho_L\Big)(0,b) = 0$ follows that

$$\begin{aligned}\Big(\theta\circ\pi_B\circ\rho_B\Big)(b) &= \langle(\phi,\theta),(0,b)\rangle - \Big((\phi,0)\circ\pi_L\circ\rho_L\Big)(0,b)\\ &= \theta(b),\end{aligned}$$

for every $b\in B$. Moreover

$$\rho(b_1b_2) = b_1\cdot\rho_B(b_2) = \theta(b_2)\rho_B(b_1), \quad (b_1,b_2\in B). \tag{22}$$

It implies that B is left $\theta-$biprojective.

Sahami in [17] introduced and studied the notion of left $\phi-$biflatness for Banach algebras. A Banach algebra A is called left $\phi-$biflat if there exists a bounded linear map $\rho_A : A \longrightarrow (A\widehat{\otimes}A)^{**}$ such that

$$\rho_A(ab) = a\cdot\rho_A(b) = \phi(b)\rho_A(a), \quad \tilde{\phi}\circ\pi_A^{**}\circ\rho_A = \phi, \quad (a,b\in A), \tag{23}$$

where $\tilde{\phi}(F) = F(\phi)$ for all $F\in A^{**}$.

Proposition 3 *Let A and B be Banach algebras. Suppose that $\theta\in\sigma(B)$ and $\phi\in\sigma(A)$. If L is left $(\phi,\theta)-$biflat, then A is left $\phi-$biflat, provided that A is unital.*

Proof Since L is left $(\phi,\theta)-$biflat, there exists a bounded linear map $\rho_L : L \longrightarrow (L\widehat{\otimes}L)^{**}$ such that

$$\rho_L(l_1l_2) = l_1\cdot\rho_L(l_2) = (\phi,\theta)(l_2)\rho_L(l_1), \quad \widetilde{(\phi,\theta)}\circ\pi_L^{**}\circ\rho_L = (\phi,\theta), \qquad (l_1,l_2\in L). \tag{24}$$

We define $\rho_A : A \longrightarrow (A\widehat{\otimes}A)^{**}$ by $\rho_A := (r_{_A}\otimes r_{_A})^{**}\circ\rho_L\circ q_{_A}$. One can see that

$$(r_A\otimes r_A)^*(\phi\circ\pi_A) = (\phi,\theta)\circ\pi_L. \tag{25}$$

It gives that

$$\begin{aligned}\langle\tilde{\phi}\circ{\pi_A}^{**}\circ\rho_A, a\rangle &= \langle\rho_A(a), {\pi_A}^*(\phi)\rangle\\ &= \langle\rho_L(a,0), (r_A\otimes r_A)^*(\phi\circ\pi_A)\rangle\\ &= \phi(a),\end{aligned}$$

for all $a \in A$. Also

$$\rho_A(a_1a_2) = (r_{_A}\otimes r_{_A})^{**}\circ\rho_L(q_{_A}(a_1a_2)) = a_1\cdot(r_{_A}\otimes r_{_A})^{**}\circ\rho_L(q_{_A}(a_2)) = a_1\cdot\rho_A(a_2) \quad (26)$$

and

$$\begin{aligned}\rho_A(a_1a_2) = (r_{_A}\otimes r_{_A})^{**}\circ\rho_L\circ q_{_A}(a_1a_2) &= (r_{_A}\otimes r_{_A})^{**}\circ\rho_L(q_{_A}(a_1)\cdot a_2)\\ &= \phi(a_2)\rho_A(a_1),\end{aligned}$$

for all a_1 and a_2 in A. Hence A is left $\phi-$biflat.

Proposition 4 *Let A and B be Banach algebras. Also let A be unital and $\psi, \theta \in \sigma(B)$. Then L is left $(0,\psi)-$biflat if and only if B is left $\psi-$biflat.*

Proof Suppose that L is left $(0,\psi)-$biflat. Then there exists a bounded linear map $\rho_L : L \longrightarrow (L\widehat{\otimes}L)^{**}$ such that

$$\rho_L(l_1l_2) = l_1\cdot\rho_L(l_2) = (0,\psi)(l_2)\rho_L(l_1), \quad \widetilde{(0,\psi)}\circ\pi_L^{**}\circ\rho_L = (0,\psi), \qquad (l_1,l_2 \in L). \quad (27)$$

We know that $\pi_B^*(\psi) = \psi\circ\pi_B$. Define $\lambda_B : B \longrightarrow (B\widehat{\otimes}B)^{**}$ by

$$\rho_B := (p_{_B}\otimes p_{_B})^{**}\circ\rho_L\circ q_{_B}. \quad (28)$$

Clearly ${\pi_L}^*((0,\psi)) = (p_B\otimes p_B)^*(\psi\circ\pi_B)$. It follows that

$$\begin{aligned}\langle\tilde{\psi}\circ{\pi_B}^{**}\circ\rho_B, b\rangle &= \langle{\pi_B}^{**}\circ\rho_B(b),\psi\rangle\\ &= \langle\rho_B(b),\psi\circ\pi_B\rangle\\ &= \langle\rho_L((0,b)),(p_{_B}\otimes p_{_B})^*(\psi\circ\pi_B)\rangle\\ &= \psi(b),\end{aligned}$$

for all $b \in B$. Also we have

$$\rho_B(b_1b_2) = b_1\cdot\rho_B(b_2) = \psi(b_2)\rho_B(b_1), \qquad (b_1,b_2 \in B). \quad (29)$$

It gives that B is left ψ-biflat.

To show the only if part, let B be left $\psi-$biflat. Then there exists a bounded linear map $\rho_B : B \longrightarrow (B\widehat{\otimes}B)^{**}$ such that

$$\rho_B(b_1b_2) = b_1\cdot\rho_B(b_2) = \psi(b_2)\rho_B(b_1), \quad \tilde{\psi}\circ\pi_B^{**}\circ\lambda_B = \psi \qquad (b_1,b_2 \in B). \quad (30)$$

One can show that

$$(S_B\otimes S_B)^*((0,\psi)\circ\pi_L) = {\pi_B}^*(\psi). \quad (31)$$

Define $\rho_L : L \longrightarrow (L\widehat{\otimes}L)^{**}$ by

$$\rho_L := (S_{_B}\otimes S_{_B})^{**}\circ\rho_B\circ p_{_B}. \quad (32)$$

Clearly ρ_L is a bounded linear map which satisfies

$$\rho_L(l_1l_2) = l_1\cdot\rho_L(l_2) = (\psi,0)(l_2)\rho_L(l_1), \quad \widetilde{(0,\psi)}\circ{\pi_L}^{**}\circ\rho_L = \psi, \quad (l_1,l_2 \in L). \quad (33)$$

It follows that L is left $(0,\psi)-$biflat.

By modifying the proof of Proposition 4 (if part), if we define

$$\rho_B = (p_{_B} \otimes p_{_B})^{**} \circ \rho_L \circ S_{_B}, \tag{34}$$

then we can show that B is left $\psi-$biflat.

3 Results

Suppose that A is a Banach algebra and $\phi \in \sigma(A)$. We remind that a Banach algebra A is left ϕ-amenable (left ϕ-contractible) if there exists an element m in A^{**} (an element m in A) such that $am = \phi(a)m$ ($am = \phi(a)m$) and $\tilde{\phi}(m) = 1$ ($\phi(m) = 1$) for all $a \in A$, respectively, see [9] and [13]. A Banach algebra A is called left character amenable (left character contractible), if A for all $\phi \in \sigma(A)$, is left ϕ-amenable (left ϕ-contractible) and A posses a bounded left approximate identity (left identity), respectively, see [13].

Example 1 We give a Lau product Banach algebra which is not left ϕ-biflat. To see this, let $C^1[0,1]$ be the set of all differentiable functions which its first derivation is continuous. Equip $C^1[0,1]$ with the point-wise multiplication and the sup-norm. Clearly $C^1[0,1]$ becomes a Banach algebra. It is known that $\sigma(C^1[0,1]) = \{\phi_t : t \in [0,1]\}$, where $\phi_t(f) = f(t)$ for all $t \in [0,1]$. We assume in contradiction that $C^1[0,1] \times_\theta C^1[0,1]$ is left $(\phi_t, \theta)-$biflat or left $(0, \phi_t)-$biflat, where $\phi_t(f) = f(t)$ for each $t \in [0,1]$. We know that the function 1 is an identity for $C^1[0,1]$. By Proposition 3 and Proposition 4 $C^1[0,1]$ is left ϕ_t-biflat. Therefore, there exists a bounded linear map $\rho : C^1[0,1] \longrightarrow (C^1[0,1]\hat{\otimes}C^1[0,1])^{**}$ such that

$$\rho_{C^1[0,1]}(fg) = f \cdot \rho_{C^1[0,1]}(g) = \phi_t(g)\rho_{C^1[0,1]}(f), \quad \tilde{\phi_t} \circ \pi^{**}_{C^1[0,1]} \circ \rho(f) = \phi_t(f) \tag{35}$$

for all $f, g \in C[0,1]$. Put $m = \pi^{**}_{C_{[0,1]}} \circ \rho(1) \in A^{**}$, we have

$$f \cdot m = f \cdot \pi^{**}_{C_{[0,1]}} \circ \rho(1) = \pi^{**}_{C_{[0,1]}} \circ \rho(f1) = \pi^{**}_{C_{[0,1]}} \circ \rho(1f) = \phi_t(f)m, \tag{36}$$

and

$$\tilde{\phi_t}(m) = \tilde{\phi_t} \circ \pi^{**}_{C_{[0,1]}} \circ \rho(1) = \phi_t(1) = 1, \tag{37}$$

for all $f \in C^1[0,1]$. It follows that $C^1[0,1]$ is left ϕ_t-amenable which is impossible by [9, Example 2.5].

The Banach algebra A is called left character biflat (left character biprojective) if A is left ϕ-biflat (left ϕ-biprojective) for each $\phi \in \sigma(A)$, respectively, see [17].

Proposition 5 *Let G be a locally compact group and let $M(G)$ be the measure algebra over G. Suppose that $\theta \in \sigma(M(G))$. Then $M(G) \times_\theta M(G)$ is left character biflat if and only if G is discrete and amenable.*

Proof Suppose that $M(G) \times_\theta M(G)$ is left character biflat. It is known that $M(G)$ has an identity. So Proposition 3 implies that $M(G)$ is left $\phi-$amenable for all $\phi \in \sigma(M(G))$ (By placing $m = \pi^{**}_{M(G)} \circ \rho(e)$, where e is the unit of $M(G)$). Since that $M(G)$ has an identity, $M(G)$ is left character amenable. Applying [11, Corollary 2.5] gives that G is discrete and amenable .
For converse, suppose that G is discrete and amenable. Then we have $M(G) = \ell^1(G)$. Thus by Johnson Theorem $\ell^1(G)$ is amenable. So [2, Corollary 2.1] finishes the proof.

Proposition 6 *Suppose that G is a locally compact group. Then $M(G) \times_\theta M(G)$ is left character biprojective if and only if G is finite.*

Proof Suppose that $M(G) \times_\theta M(G)$ is left character biprojective. Then by Proposition 1, $M(G)$ is left character biprojective ($M(G)$ is unital). One can easily see that $M(G)$ is left $\phi-$contractible for all $\phi \in \sigma(M(G))$. Since $M(G)$ is unital, it follows that $M(G)$ is left character contractible. From [13, Corollary 6.2], we have G is a finite group.
Converse is clear.

It is well-known that the Fourier algebra $A(G)$ over a locally compact group G is a commutative Banach algebra. Also, $\sigma(A(G)) = \{\phi_g : g \in G\}$, where $\phi_g(f) = f(g)$, see [14].

Theorem 1 *Suppose that G is a locally compact group. Then $M(G) \times_\theta A(G)$ is left character biprojective if and only if G is a finite group.*

Proof Similar to the proof of previous Proposition.

Suppose that $N_\vee$ is the semigroup N (the natural numbers) with products $m \vee n = \max\{m, n\}$. Consider $\ell^1(N_\vee)$ with convolution product. We denote δ_n for the *point mass* at $\{n\}$. For every $n \in N$, we consider a homomorphism $\phi_n : \ell^1(N_\vee) \longrightarrow C$ with the formula $\phi_n\left(\sum_{i=1}^{\infty} \alpha_i \delta_i\right) = \sum_{i=1}^{n} \alpha_i$ for each $n \in N \cup \{\infty\}$. It is known that

$$\sigma(\ell^1(N_\vee)) = \{\phi_n : n \in N \cup \{\infty\}\} \tag{38}$$

We write $\phi_{N_\vee} = \phi_\infty$ for the *augmentation character*, see [4].

Theorem 2 *The Banach algebra $\ell^1(N_\vee) \times_\theta \ell^1(N_\vee)$ is neither $(\phi_{N_\vee}, \theta)$-biprojective nor $(0, \phi_{N_\vee})$-biprojective, where $\phi_{N_\vee}$ is the augmentation character on $\ell^1(N_\vee)$.*

Proof We assume conversely that $\ell^1(N_\vee) \times_\theta \ell^1(N_\vee)$ is either left $(\phi_{N_\vee}, \theta)$-biprojective or left $(0, \phi_{N_\vee})$-biprojective. Since $N_\vee$ is unital, $\ell^1(N_\vee)$ has an identity. By Proposition 1 and Proposition 2 $\ell^1(N_\vee)$ is left $\phi_{N_\vee}$-biprojective. The existence of a unit δ_1 implies that $\ell^1(N_\vee)$ is left $\phi_{N_\vee}$-contractible. Now we claim that $\ell^1(N_\vee)$ is left ϕ_n-contractible for all $n \in N$. To see this define

$m_n = \delta_n - \delta_{n+1} \in \ell^1(N_\vee)$. Let $a = \sum_{n=1}^{\infty} a_n\delta_n \in \ell^1(N_\vee)$, where (a_n) is a sequence in C such that $\sum_{n=1}^{\infty} |a_n| < \infty$. Consider

$$am_n = a(\delta_n - \delta_{n+1}) = \sum_{n=1}^{\infty} a_n\delta_n(\delta_n - \delta_{n+1}) = \phi_n(a)(\delta_n - \delta_{n+1}) = \phi_n(a)m_n \tag{39}$$

and

$$\phi_n(m_n) = \phi_n(\delta_n - \delta_{n+1}) = \phi_n(\delta_n) - \phi_n(\delta_{n+1}) = 1,$$

for every $a \in \ell^1(N_\vee)$. Thus $\ell^1(N_\vee)$ is character contractible. Applying [5, Corollary 2.2] follows that $\sigma(\ell^1(N_\vee)) = N_\vee \cup \{\infty\}$ is discrete with respect to the w^*-topology. Using the Gelfand representation theorem, we have $\sigma(\ell^1(N_\vee)) = N_\vee \cup \{\infty\}$ is compact, so is finite which is a contradiction.

Example 2 Suppose that $A = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : \ a, b, c \in C\}$ be a matrix algebra. With matrix operation and ℓ^1-norm A becomes a Banach algebra. Define $\phi : A \longrightarrow C$ by $\phi(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = c$. It is easy to see that is a character on A. We claim that $A \times_\theta A$ is neither (ϕ, θ)- biflat nor $(0, \phi)-$biflat, where $\theta \in \sigma(A)$. Suppose in contradiction that $A \times_\theta A$ is either (ϕ, θ)-biflat or $(0, \phi)$-biflat. Since A is unital, by Proposition 3 and Proposition 4 A is left ϕ-biflat. Since A is unital, it is easy to see that A is left ϕ-amenable. Set $J := \{\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} : \ b, d \in C\}$ and $\phi_{|_J} \neq 0$. It is clear that J is a closed ideal of A. Since A is left ϕ-amenable, by [9, Lemma 3.1] we have that J is $\phi_{|_J}-$amenable. Now [9, Theorem 1.4] follows that, there exists a bounded net (u_α) in J such that $ju_\alpha - \phi(j)u_\alpha \longrightarrow 0$ and $\phi(u_\alpha) = 1$ for all $j \in J$. Let $j = \begin{pmatrix} 0 & j_1 \\ 0 & j_2 \end{pmatrix}$ and $u_\alpha = \begin{pmatrix} 0 & w_\alpha \\ 0 & v_\alpha \end{pmatrix}$, for some $j_1, j_2, w_\alpha, v_\alpha \in C$. Thus,

$$ju_\alpha - \phi(j)u_\alpha = \begin{pmatrix} 0 & j_1w_\alpha \\ 0 & j_2v_\alpha \end{pmatrix} - \begin{pmatrix} 0 & j_2w_\alpha \\ 0 & j_2v_\alpha \end{pmatrix} \longrightarrow 0. \tag{40}$$

It gives that $j_1v_\alpha - j_2w_\alpha \longrightarrow 0$. If we put $j_1 = 1$ and $j_2 = 0$, then we have $v_\alpha \to 0$ which contradicts with $\phi(u_\alpha) = v_\alpha = 1$.

Acknowledgments. The authors would like to thank the anonymous reviewers for their valuable comments and suggestions. The first author is thankful to Ilam university for its support.

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