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CLOSE-TO-REGULARITY OF BOUNDED TRI-LINEAR MAPS

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Abstract Let $f : X \times Y \times Z \longrightarrow W$ be a bounded tri-linear map on normed spaces. We say that f is close-to-regular when $f^{t*****s} = f^{s*****t}$ and we say that f is Aron-Berner regular when all natural extensions are equal. In this manuscript, we give a simple criterion for the close-to-regularity of tri-linear maps.

Keywords Arens regularity · Aron-Berner regular · Close-to-regular

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1 Introduction

Richard Arens showed in [1] that a bounded bilinear map $m : X \times Y \longrightarrow Z$ on normed spaces, has two natural different extensions m^{***}, m^{r***r} from $X^{**} \times Y^{**}$ into Z^{**}. When these extensions are equal, m is said to be Arens regular. For a discussion of Arens regularity for Banach algebras and bounded bilinear maps, see [2], [7], [9], [11] and [12]. For example, every C^*-algebra is Arens regular, see [6].

Let X, Y, Z and W be normed spaces and $f : X \times Y \times Z \longrightarrow W$ be a bounded tri-linear mapping. The natural extensions of f are as following:

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1. $f^*: W^* \times X \times Y \longrightarrow Z^*$, given by $\langle f^*(w^*, x, y), z\rangle = \langle w^*, f(x, y, z)\rangle$ where $x \in X, y \in Y, z \in Z, w^* \in W^*$.
2. $f^{**} = (f^*)^* : Z^{**} \times W^* \times X \longrightarrow Y^*$, given by $\langle f^{**}(z^{**}, w^*, x), y\rangle = \langle z^{**}, f^*(w^*, x, y)$ where $x \in X, y \in Y, z^{**} \in Z^{**}, w^* \in W^*$.
3. $f^{***} = (f^{**})^* : Y^{**} \times Z^{**} \times W^* \longrightarrow X^*$, given by $\langle f^{***}(y^{**}, z^{**}, w^*), x\rangle = \langle y^{**}, f^{**}(z^{**}, w^*, x)\rangle$ where $x \in X, y^{**} \in Y^{**}, z^{**} \in Z^{**}, w^* \in W^*$.
4. $f^{****} = (f^{***})^* : X^{**} \times Y^{**} \times Z^{**} \longrightarrow W^{**}$, given by $\langle f^{****}(x^{**}, y^{**}, z^{**}), w^*\rangle = \langle x^{**}, f^{***}(y^{**}, z^{**}, w^*)\rangle$ where $x^{**} \in X^{**}, y^{**} \in Y^{**}, z^{**} \in Z^{**}, w^* \in W^*$.

The bounded tri-linear map f^{****} is the extension of f such that the maps

$$\begin{aligned}
&x^{**} \longrightarrow f^{****}(x^{**}, y^{**}, z^{**}) : X^{**} \longrightarrow W^{**},\\
&y^{**} \longrightarrow f^{****}(x, y^{**}, z^{**}) : Y^{**} \longrightarrow W^{**},\\
&z^{**} \longrightarrow f^{****}(x, y, z^{**}) : Z^{**} \longrightarrow W^{**},
\end{aligned}$$

are weak$^*-$weak* continuous for each $x \in X, y \in Y, x^{**} \in X^{**}, y^{**} \in Y^{**}$ and $z^{**} \in Z^{**}$. Now let

$$\begin{aligned}
&f^i : Y \times X \times Z \longrightarrow W : f^i(y, x, z) = f(x, y, z),\\
&f^j : X \times Z \times Y \longrightarrow W : f^j(x, z, y) = f(x, y, z),\\
&f^r : Z \times Y \times X \longrightarrow W : f^r(z, y, x) = f(x, y, z),\\
&f^t : Z \times X \times Y \longrightarrow W : f^t(z, x, y) = f(x, y, z),\\
&f^s : Y \times Z \times X \longrightarrow W : f^s(y, z, x) = f(x, y, z),
\end{aligned}$$

be the flip maps of f. The flip maps of f are bounded tri-linear maps. It is easily seen that $f^{i****i}, f^{j****j}, f^{r****r}, f^{t****s}$ and f^{s****t} are natural extensions of f such that bounded linear operators

$$\begin{aligned}
&x^{**} \longrightarrow f^{i****i}(x^{**}, y, z^{**}) : X^{**} \longrightarrow W^{**},\\
&y^{**} \longrightarrow f^{i****i}(x^{**}, y^{**}, z^{**}) : Y^{**} \longrightarrow W^{**},\\
&z^{**} \longrightarrow f^{i****i}(x, y, z^{**}) : Z^{**} \longrightarrow W^{**},\\
&x^{**} \longrightarrow f^{j****j}(x^{**}, y, z^{**}) : X^{**} \longrightarrow W^{**},\\
&y^{**} \longrightarrow f^{j****j}(x, y^{**}, z) : Y^{**} \longrightarrow W^{**},\\
&z^{**} \longrightarrow f^{j****j}(x, y^{**}, z^{**}) : Z^{**} \longrightarrow W^{**},\\
&x^{**} \longrightarrow f^{r****r}(x^{**}, y, z) : X^{**} \longrightarrow W^{**},\\
&y^{**} \longrightarrow f^{r****r}(x^{**}, y^{**}, z) : Y^{**} \longrightarrow W^{**},\\
&z^{**} \longrightarrow f^{r****r}(x^{**}, y^{**}, z^{**}) : Z^{**} \longrightarrow W^{**},\\
&x^{**} \longrightarrow f^{t****s}(x^{**}, y^{**}, z) : X^{**} \longrightarrow W^{**},\\
&y^{**} \longrightarrow f^{t****s}(x, y^{**}, z) : Y^{**} \longrightarrow W^{**},\\
&z^{**} \longrightarrow f^{t****s}(x^{**}, y^{**}, z^{**}) : Z^{**} \longrightarrow W^{**},\\
&x^{**} \longrightarrow f^{s****t}(x^{**}, y, z) : X^{**} \longrightarrow W^{**},\\
&y^{**} \longrightarrow f^{s****t}(x^{**}, y^{**}, z^{**}) : Y^{**} \longrightarrow W^{**},\\
&z^{**} \longrightarrow f^{s****t}(x^{**}, y, z^{**}) : Z^{**} \longrightarrow W^{**},
\end{aligned}$$

are weak*−weak* continuous for each $x \in X, y \in Y, z \in Z, x^{**} \in X^{**}, y^{**} \in Y^{**}$ and $z^{**} \in Z^{**}$. For natural extensions of f we have

1. $f^{i*****i}(x^{**}, y^{**}, z^{**}) = w^* - \lim_\beta w^* - \lim_\alpha w^* - \lim_\gamma f(x_\alpha, y_\beta, z_\gamma)$,
2. $f^{j*****j}(x^{**}, y^{**}, z^{**}) = w^* - \lim_\alpha w^* - \lim_\gamma w^* - \lim_\beta f(x_\alpha, y_\beta, z_\gamma)$,
3. $f^{r*****r}(x^{**}, y^{**}, z^{**}) = w^* - \lim_\gamma w^* - \lim_\beta w^* - \lim_\alpha f(x_\alpha, y_\beta, z_\gamma)$,
4. $f^{****}(x^{**}, y^{**}, z^{**}) = w^* - \lim_\alpha w^* - \lim_\beta w^* - \lim_\gamma f(x_\alpha, y_\beta, z_\gamma)$,
5. $f^{t*****s}(x^{**}, y^{**}, z^{**}) = w^* - \lim_\gamma w^* - \lim_\alpha w^* - \lim_\beta f(x_\alpha, y_\beta, z_\gamma)$,
6. $f^{s*****t}(x^{**}, y^{**}, z^{**}) = w^* - \lim_\beta w^* - \lim_\gamma w^* - \lim_\alpha f(x_\alpha, y_\beta, z_\gamma)$,

where $\{x_\alpha\}, \{y_\beta\}$ and $\{z_\gamma\}$ are nets in X, Y and Z which converge to $x^{**} \in X^{**}, y^{**} \in Y^{**}$ and $z^{**} \in Z^{**}$ in the w^*−topologies, respectively. More information about these maps can be found in [10] and [13].

Definition 1 A bounded tri-linear map f is said to be close-to-regular if $f^{t*****s} = f^{s*****t}$. It is obvious that f is close-to-regular if and only if $f^{s******s} = f^{t******j}$ on $Y^{**} \times Z^{**} \times W^{***}$.

Definition 2 A bounded tri-linear map f is said to be Aron-Berner regular when all natural extensions are equal, that is, $f^{i*****i} = f^{j*****j} = f^{r*****r} = f^{****} = f^{t*****s} = f^{s*****t}$ holds. For example see [10], see also [3], [4] and [5]. If f is Aron-Berner regular, then trivially f is close-to-regular.

Throughout the article, we usually identify a normed space with its canonical image in its second dual.

2 Close-to-regular maps

We commence with the following theorem for close-to-regular maps.

Theorem 1 *For a bounded tri-linear map $f : X \times Y \times Z \longrightarrow W$ the following statements are equivalent:*

1. *f is close-to-regular.*
2. *$f^{s****t*}(Y^{**}, W^*, Z) \subseteq X^*$ and $f^{s*******}(X^{**}, W^*, Y^{**}) \subseteq Z^*$.*
3. *$f^{t*****}(W^*, Z^{**}, X^{**}) \subseteq Y^*$.*

Proof Suppos $\{x_\alpha\}, \{y_\beta\}$ and $\{z_\gamma\}$ are nets in X, Y and Z which converge to $x^{**} \in X^{**}, y^{**} \in Y^{**}$ and $z^{**} \in Z^{**}$ in the w^*−topologies, respectively.

(1) ⇒ (2), if f is close-to-regular, then $f^{t*****s} = f^{s*****t}$. For every $x^{**} \in X^{**}, y^{**} \in Y^{**}, z \in Z$ and $w^* \in W^*$ we have

$$\begin{aligned}&\langle\ f^{s****t*}(y^{**}, w^*, z), x^{**}\rangle = \langle y^{**}, f^{s****}(z, x^{**}, w^*)\rangle\\ &= \langle f^{s*****t}(x^{**}, y^{**}, z), w^*\rangle = \langle f^{t*****s}(x^{**}, y^{**}, z), w^*\rangle\\ &= \langle f^{t****}(z, x^{**}, y^{**}), w^*\rangle = \langle f^{t**}(y^{**}, w^*, z), x^{**}\rangle.\end{aligned}$$

Therefore $f^{s****t*}(y^{**}, w^*, z) = f^{t**}(y^{**}, w^*, z) \in X^*$, follows that

$$f^{s****t*}(Y^{**}, W^*, Z) \subseteq X^*.$$

In the other hand,

$$\begin{aligned}
&\langle\ f^{s******}(x^{**}, w^*, y^{**}), z^{**}\rangle = \langle w^*, f^{s****}(y^{**}, z^{**}, x^{**})\rangle \\
&= \langle w^*, f^{s*****t}(x^{**}, y^{**}, z^{**})\rangle = \langle w^*, f^{t*****s}(x^{**}, y^{**}, z^{**})\rangle \\
&= \langle w^*, f^{t****}(z^{**}, x^{**}, y^{**})\rangle = \langle z^{**}, f^{t***}(x^{**}, y^{**}, w^*)\rangle.
\end{aligned}$$

Since the $f^{t***}(x^{**}, y^{**}, w^*) \in z^*$, thus $f^{s******}(X^{**}, W^*, Y^{**}) \subseteq Z^*$, as claimed.

(2) $\Rightarrow$ (3), if (2) holds then

$$\begin{aligned}
&\langle\ f^{t*****}(w^*, z^{**}, x^{**}), y^{**}\rangle = \lim_\gamma \lim_\alpha \lim_\beta \langle f(x_\alpha, y_\beta, z_\gamma), w^*\rangle \\
&= \lim_\gamma \lim_\alpha \lim_\beta \langle w^*, f^s(y_\beta, z_\gamma, x_\alpha)\rangle = \lim_\gamma \lim_\alpha \lim_\beta \langle f^{s***}(z_\gamma, x_\alpha, w^*), y_\beta\rangle \\
&= \lim_\gamma \lim_\alpha \langle y^{**}, f^{s****t}(w^*, z_\gamma, x_\alpha)\rangle = \lim_\gamma \lim_\alpha \langle f^{s****t*}(y^{**}, w^*, z_\gamma), x_\alpha\rangle \\
&= \lim_\gamma \langle f^{s****t*}(y^{**}, w^*, z_\gamma), x^{**}\rangle = \lim_\gamma \langle y^{**}, f^{s****t}(w^*, z_\gamma, x^{**})\rangle \\
&= \lim_\gamma \langle y^{**}, f^{s****}(z_\gamma, x^{**}, w^*)\rangle = \lim_\gamma \langle f^{s******}(x^{**}, w^*, y^{**}), z_\gamma\rangle \\
&= \langle f^{s******}(x^{**}, w^*, y^{**}), z^{**}\rangle = \langle f^{s****}(z^{**}, x^{**}, w^*), y^{**}\rangle \\
&= \langle f^{s****t}(w^*, z^{**}, x^{**}), y^{**}\rangle.
\end{aligned}$$

Since $f^{s****t}(w^*, z^{**}, x^{**}) \in Y^*$, thus (3) holds.

(3) $\Rightarrow$ (1), let $f^{t*****}(W^*, Z^{**}, X^{**}) \subseteq Y^*$. Then for every $w^* \in W^*$ we have,

$$\begin{aligned}
&\langle\ f^{s*****t}(x^{**}, y^{**}, z^{**}), w^*\rangle = \lim_\beta \lim_\gamma \lim_\alpha \langle f(x_\alpha, y_\beta, z_\gamma), w^*\rangle \\
&= \lim_\beta \lim_\gamma \lim_\alpha \langle w^*, f^t(z_\gamma, x_\alpha, y_\beta\rangle = \lim_\beta \lim_\gamma \lim_\alpha \langle f^{t*}(w^*, z_\gamma, x_\alpha), y_\beta\rangle \\
&= \lim_\beta \lim_\gamma \lim_\alpha \langle f^{t**}(y_\beta, w^*, z_\gamma), x_\alpha\rangle = \lim_\beta \lim_\gamma \langle x^{**}, f^{t**}(y_\beta, w^*, z_\gamma)\rangle \\
&= \lim_\beta \lim_\gamma \langle f^{t***}(x^{**}, y_\beta, w^*), z_\gamma\rangle = \lim_\beta \langle z^{**}, f^{t***}(x^{**}, y_\beta, w^*)\rangle \\
&= \lim_\beta \langle f^{t****}(z^{**}, x^{**}, y_\beta), w^*\rangle = \lim_\beta \langle f^{t*****}(w^*, z^{**}, x^{**}), y_\beta\rangle \\
&= \langle f^{t*****}(w^*, z^{**}, x^{**}), y^{**}\rangle = \langle f^{t*****s}(x^{**}, y^{**}, z^{**}), w^*\rangle.
\end{aligned}$$

It follows that f is close-to-regular and this completes the proof.

As an immediate consequence of Theorem 1, we deduce the next result.

Corollary 1 *Let $f : X \times Y \times Z \longrightarrow W$ be a bounded tri-linear mapping.*

1. *If Y is reflexive, then f is close-to-regular.*
2. *If X and Z are reflexive, then f is close-to-regular.*

Example 1 Let G be a compact group. Then $L^p(G)$ for $p > 1$ is a reflexive Banach algebra. So the bounded tri-linear mapping

$$f : L^p(G) \times L^p(G) \times L^p(G) \longrightarrow L^p(G)$$

defined by $f(k, g, h) = k*g*h$ is close-to-regular, where $(k*g)(x) = \int_G k(y)g(y^{-1}x)dy$ for every k, g and $h \in L^p(G)$, see [8].

Theorem 2 *Let $f : X \times Y \times Z \longrightarrow W$ be a bounded tri-linear map. Then,*

1. *f^r is close-to-regular if and only if $f^{i*****i} = f^{j****j}$.*
2. *f^i is close-to-regular if and only if $f^{j****j} = f^{r*****r}$.*
3. *f^j is close-to-regular if and only if $f^{i*****i} = f^{r*****r}$.*
4. *f^t is close-to-regular if and only if $f^{s*****t} = f^{****}$.*
5. *f^s is close-to-regular if and only if $f^{t*****s} = f^{****}$.*

Proof We prove only (1), the other parts have the same argument. Let $x^{**} \in X^{**}, y^{**} \in Y^{**}, z^{**} \in Z^{**}$ and $w^* \in W^*$ and let $\{x_\alpha\}, \{y_\beta\}$ and $\{z_\gamma\}$ be nets in X, Y and Z which converge to x^{**}, y^{**} and z^{**} in the w^*-topologies, respectively. Then we have

$$\begin{aligned}\langle f^{i*****i}(x^{**}, y^{**}, z^{**}), w^*\rangle &= \lim_\beta \lim_\alpha \lim_\gamma \langle f(x_\alpha, y_\beta, z_\gamma), w^*\rangle \\ &= \lim_\beta \lim_\alpha \lim_\gamma \langle f^r(z_\gamma, y_\beta, x_\alpha), w^*\rangle \\ &= \langle f^{rs*****t}(z^{**}, y^{**}, x^{**}), w^*\rangle.\end{aligned}$$

Therefore $f^{i*****i} = f^{rs*****t}$. In the other hand

$$\begin{aligned}\langle f^{j****j}(x^{**}, y^{**}, z^{**}), w^*\rangle &= \lim_\alpha \lim_\gamma \lim_\beta \langle f(x_\alpha, y_\beta, z_\gamma), w^*\rangle \\ &= \lim_\alpha \lim_\gamma \lim_\beta \langle f^r(z_\gamma, y_\beta, x_\alpha), w^*\rangle \\ &= \langle f^{rt*****s}(z^{**}, y^{**}, x^{**}), w^*\rangle.\end{aligned}$$

Thus $f^{j****j} = f^{rt*****s}$ and this completes the proof.

Another proof: Since the $f^{rt} = f^j = f^{sr}$ and $f^{rs} = f^i = f^{tr}$, thus f^r is close-to-regular if and only if

$$f^{rt*****s} = f^{rs*****t} \Leftrightarrow f^{rt*****sr} = f^{rs*****tr} \Leftrightarrow f^{j****j} = f^{i*****i}.$$

As immediate consequences of Theorem 2, we have the next corollaries.

Corollary 2 *If f is Aron-Berner regular, then f^i, f^j, f^r, f^t and f^s are close-to-regular.*

Corollary 3 *If f^s and f^t are close-to-regular, then f is close-to-regular.*

Theorem 3 *Let $f : X \times Y \times Z \longrightarrow W$ and $g : X \times S \times Z \longrightarrow W$ be bounded tri-linear mappings and let $h : Y \longrightarrow S$ be a bounded linear mapping such that $f(x, y, z) = g(x, h(y), z)$, for every $x \in X, y \in Y$ and $z \in Z$. If h is weakly compact, then f is close-to-regular.*

Proof Suppos $\{x_\alpha\}, \{y_\beta\}$ and $\{z_\gamma\}$ are nets in X, Y and Z which converge to $x^{**} \in X^{**}, y^{**} \in Y^{**}$ and $z^{**} \in Z^{**}$ in the w^*-topologies, respectively. Then a direct verification reveals that

$$f^{t*****s}(x^{**}, y^{**}, z^{**}) = g^{t*****s}(x^{**}, h^{**}(y^{**}), z^{**}).$$

Then for each $y^{**} \in Y^{**}$ we have

$$\begin{aligned}
\langle f^{t*****}(w^*, z^{**}, x^{**}), y^{**}\rangle &= \langle w^*, f^{t****}(z^{**}, x^{**}, y^{**})\rangle \\
&= \langle w^*, f^{t*****s}(x^{**}, y^{**}, z^{**})\rangle \\
&= \langle w^*, g^{t*****s}(x^{**}, h^{**}(y^{**}), z^{**})\rangle \\
&= \langle w^*, g^{t****}(z^{**}, x^{**}, h^{**}(y^{**}))\rangle \\
&= \langle g^{t*****}(w^*, z^{**}, x^{**}), h^{**}(y^{**})\rangle \\
&= \langle h^{***}(g^{t*****}(w^*, z^{**}, x^{**})), y^{**}\rangle.
\end{aligned}$$

Therefore $f^{t*****}(w^*, z^{**}, x^{**}) = h^{***}(g^{t*****}(w^*, z^{**}, x^{**}))$. The weak compactness of h implies that of h^*, from which we have $h^{***}(S^{***}) \subseteq Y^*$. In particular,

$$h^{***}(g^{t*****}(W^*, Z^{**}, X^{**})) \subseteq Y^*,$$

thus we deduce $f^{t*****}(W^*, Z^{**}, X^{**}) \subseteq Y^*$. It follows that f is close-to-regular and this completes the proof.

If Y or S is reflexive, then every bounded linear mapping $h : Y \longrightarrow S$ is weakly compact. Thus we give the next result.

Corollary 4 *Let $f : X \times Y \times Z \longrightarrow W$ and $g : X \times S \times Z \longrightarrow W$ be bounded tri-linear mappings and let $h : Y \longrightarrow S$ be a bounded linear mapping such that $f(x, y, z) = g(x, h(y), z)$, for every $x \in X, y \in Y$ and $z \in Z$. If S is reflexive, then f is close-to-regular.*

Theorem 4 *Let $f : X \times Y \times Z \longrightarrow W$ be bounded tri-linear mapping. If $f^{****t**s} = f^{t**s*****}$ and $f^{*****s**t} = f^{s**t****}$. Then f is close-to-regular*

Proof Using the equality $f^{*****s**t} = f^{s**t****}$, a standard argument applies to show that $f^{****} = f^{s*****t}$. In the other hand, the equality $f^{****t**s} = f^{t**s*****}$ impleas that $f^{****} = f^{t*****s}$. Therefore $f^{s*****t} = f^{t*****s}$, as claimed.

Note that for theorem 4 the converse is not true.

3 Conclusion

In this manuscript, the authors investigated Aron-Berner regularity and close-to-regularity of bounded tri-linear maps. In Section 2 some necessary and sufficient conditions on tri-linear maps which guarantee their close-to- regularity are provided.

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