Global Analysis and Discrete Mathematics
Volume 6, Issue 1, pp. 73–81
ISSN: 2476-5341

Some Remarks On The Varieties of Groups

Homayoon Arabyani

Received: 27 February 2021 / Accepted: 16 April 2021

Abstract Let $\mathcal{V}$ be a variety of groups defined by a set V of laws. Let (N,G) be a pair of groups in which N is a normal subgroup of G. We define the lower and upper $\mathcal{V}$-marginal series of the pair (N,G) and prove some results on $\mathcal{V}$-nilpotent pairs of groups. Moreover, we extend some properties of the Baer-invariant and isologism of a pair of groups.

Keywords Pair of groups · Baer-invariant · Isologism.

Mathematics Subject Classification (2010) 20D15 . 20E99

1 Introduction and Preliminaries

Let F be a free group on a set $\{x_1, x_2, \ldots\}$, V be a non-empty subset of F and $\mathcal{V}$ be a variety of groups defined by a set V of laws. For a pair of groups (N,G) in which N is a normal subgroup of G, we define

$$V(N,G) = \langle v(g_1,\ldots,g_in,\ldots,g_r)v(g_1,\ldots,g_r)^{-1} : v \in V,\, n \in N,\, g_i \in G,\, 1 \leq i \leq r\rangle,$$

and

$$V^*(N,G) = \{n \in N : v(g_1,\ldots,g_in,\ldots,g_r) = v(g_1,\ldots,g_r), \forall v \in V,\, g_i \in G,\, 1 \leq i \leq r\}.$$

In particular, if $N = G$, then $V(N,G) = V(G)$ and $V^*(N,G) = V^*(G)$ are ordinary verbal and marginal subgroups of G (see [5,10,15]).

If $\mathcal{V}$ is the variety of nilpotent groups of class at most n, then

$$V^*(N,G) = Z_n(N,G) \quad and \quad V(N,G) = \gamma_{n+1}(N,G),$$

H. Arabyani
Department of Mathematics, Neyshabur Branch, Islamic Azad University, Neyshabur, Iran;
Tel.: +123-45-678910
Fax: +123-45-678910
E-mail: arabyani.h@gmail.com, h.arabyani@iau-neyshabur.ac.ir

where $\gamma_1(N,G) = N$ and $\gamma_{n+1}(N,G) = [\gamma_n(N,G), G]$ for all $n \geq 1$. Moreover, $Z_0(N,G) = 1$ and for all $n \geq 1$,

$$\frac{Z_n(N,G)}{Z_{n-1}(N,G)} = Z(\frac{N}{Z_{n-1}(N,G)}, \frac{G}{Z_{n-1}(N,G)}). \tag{1}$$

Let $\mathcal{V}$ and $\mathcal{W}$ be two arbitrary varieties of groups defined by the sets of laws V and W, respectively, and G a group with a free presentation

$$1 \to R \to F \to G \to 1.$$

Then, we define the Baer-invariant of G with respect to two varieties $\mathcal{V}$ and $\mathcal{W}$ as follows

$$\mathcal{WVM}(G) = \frac{\big(R \cap W(F)\big)\big(R \cap V(F)\big)}{\big(R \cap W(F)\big)[RV^*F]},$$

where $[RV^*F]$ is the least normal subgroup T say, of F contained in R so that $R/T \subseteq V^*(F/T)$ (see [9] for more information). In this paper, we are going to generalize some results of [3, 7, 14, 15].

2 Some inequalities on the Baer-invariant of a pair of groups

In this section, we extend some results of [3, 14, 15]. To this end, we define the lower and the upper $\mathcal{V}$-marginal series of a pair (N,G). Moreover, we prove some results on nilpotent pairs of groups with respect to a variety of groups. Also, we prove some properties on the Baer-invariant of a pair of groups.

For a pair of groups (N,G) put $V_0(N,G) = N$ and define

$$V_i(N,G) = V(V_{i-1}(N,G), G).$$

Then

$$N = V_0(N,G) \supseteq V_1(N,G) \supseteq \ldots \supseteq V_n(N,G) \supseteq \ldots$$

is the called the lower $\mathcal{V}$-marginal series of N in G. Similarly, we define the upper $\mathcal{V}$-marginal series of N in G, by setting

$$V_0^*(N,G) = \langle e\rangle, \quad \frac{V_i^*(N,G)}{V_{i-1}^*(N,G)} = V^*\left(\frac{N}{V_{i-1}^*(N,G)}, \frac{G}{V_{i-1}^*(N,G)}\right).$$

The pair (N,G) of groups is said to be $\mathcal{V}$-nilpotent pair, if $V_n(N,G) = \langle e\rangle$ for some positive integer n (see [2], section 3).

Theorem 1 *Let $\mathcal{V}$ be a variety of groups. If (N,G) is a $\mathcal{V}$-nilpotent pair of groups and M is a non-trivial normal subgroup of G such that $M \cap N \neq \langle e\rangle$, then $M \cap V^*(N,G) \neq \langle e\rangle$.*

Proof Since (N, G) is $\mathcal{V}$-nilpotent, then there exists a positive integer c such that $V_c^*(N, G) = N$. Let i be the least integer such that $M \cap V_i^*(N, G) \neq \langle e \rangle$. We have

$$V\left(M \cap V_i^*(N, G), G\right) \leq M \cap V_{i-1}^*(N, G) = \langle e \rangle$$

and

$$M \cap V_i^*(N, G) \leq M \cap V^*(N, G)$$

Hence,

$$M \cap V^*(N, G) = M \cap V_i^*(N, G) \neq \langle e \rangle.$$

The following corollary is an immediate result of Theorem 1.

Corollary 1 *If (N, G) is a $\mathcal{V}$-nilpotent pair of groups with $N \neq \langle e \rangle$, then $V^*(N, G) \neq \langle e \rangle$.*

Theorem 2 *Let $\mathcal{V}$ be a variety of groups and (N, G) be a $\mathcal{V}$-nilpotent pair of groups, then every maximal subgroup of G which does not contain N, is normal.*

Proof By the assumption there exists a positive integer c such that

$$V_c^*(N, G) = N,$$

thus for every maximal subgroup M of G, the following series is a subnormal series for M

$$M \trianglelefteq MV^*(N, G) \trianglelefteq MV_2^*(N, G) \trianglelefteq ... \trianglelefteq MV_c^*(N, G) = MN = G.$$

If $c = 1$, then $M \trianglelefteq G$. Since M is a maximal subgroup of G, then there exists a least positive integer j where $MV_j^*(N, G) = G$. Then, $MV_{j-1}^*(N, G) = M$, and so $M \trianglelefteq G$.

Theorem 3 *Let $\mathcal{V}$ be a variety of groups. If (N, G) is a pair of groups such that $K \leq V^*(N, G)$ and $(N/K, G/K)$ is a $\mathcal{V}$-nilpotent pair of groups, then (N, G) is a $\mathcal{V}$-nilpotent pair.*

Proof Since $(N/K, G/K)$ is a $\mathcal{V}$-nilpotent pair of groups. So, there exist a normal series as

$$1 = \frac{N_1}{K} \leq \frac{N_2}{K} \leq ... \leq \frac{N_n}{K} = \frac{N}{K}$$

such that

$$\frac{N_{i+1}/N}{N_i/N} \leq V^*\left(\frac{N/K}{N_i/K}, \frac{G/K}{N_i/K}\right).$$

Now, we have $N_{i+1}/N_i \leq V^*\left(N/N_i, G/N_i\right)$. Hence, we obtain the following normal series.

$$1 = N_0 \leq N_1 \leq ... \leq N_n = N.$$

Thus, the pair (N, G) is a $\mathcal{V}$-nilpotent pair.

Theorem 4 *Let $\mathcal{V}$ be a variety of groups. If (N,G) is a $\mathcal{V}$-nilpotent pair and $K \trianglelefteq N$ such that $|K| = p^n$. Then $K \leq V_n^*(N,G)$.*

Proof We prove the result by using induction. Let $n = 1$, then

$$\langle e\rangle \neq K \cap V^*(N,G) \leq K,$$

so $|K \cap V^*(N,G)| = p = |K|$, hence $K \cap V^*(N,G) = K$ and so, $K \leq V^*(N,G)$. Let the result holds for every number less than n and $M = K \cap V^*(N,G) \neq \langle e\rangle$, then $|K/M| = p^m$ where $m < n$. Now we have

$$\frac{K}{M} = \frac{K}{K \cap V^*(N,G)} \cong \frac{KV^*(N,G)}{V^*(N,G)}.$$

By induction hypothesis, we have

$$\begin{aligned}\frac{KV^*(N,G)}{V^*(N,G)} &\leq V_m^*\left(\frac{N}{V^*(N,G)}, \frac{G}{V^*(N,G)}\right)\\ &= \frac{V_{m+1}^*(N,G)}{V^*(N,G)}.\end{aligned}$$

Thus, $K \leq V_{m+1}^*(N,G) \leq V_n^*(N,G)$.

Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups defined by the sets of laws V and W, respectively and G be a finite group in $\mathcal{W}$ with a free presentation $1 \to R \to F \to G \to 1$. If N is a normal subgroup of G and S is a normal subgroup of F such that $N \cong S/R$, then the Baer-invariant of the pair (N,G) with respect to $\mathcal{V}$ and $\mathcal{W}$ is defined as

$$\mathcal{WVM}(N,G) = \frac{W(F)\big(R \cap [SV^*F]\big)}{W(F)[RV^*F]},$$

where $[XV^*Y] = V(X,Y)$. One can check that $\mathcal{WVM}(N,G)$ is abelian and independent of the choice of the free presentation of G. In the special case where $\mathcal{W}$ is the variety of all groups, then

$$\mathcal{WVM}(N,G) = \mathcal{VM}(N,G)$$

is the Baer-invariant of the pair (N,G) with respect to the variety $\mathcal{V}$ (see [2, 8,11–14] for more information).
In the following theorems we generalize some results of M.R. Rismanchian and M. Araskhan in [14] .

Theorem 5 *Let (N,G) be a pair of finite groups and $1 \to R \to F \to G \to 1$ be a free presentation of G such that $N \cong S/R$ for a normal subgroup S of F. If N is a subgroup $\mathcal{V}$-nilpotent of G of class $c \geq 2$, then*

(i) $|V_{c-1}(N,G)||\mathcal{WVM}(N,G)| = \Big|\mathcal{WVM}\big(\frac{N}{V_{c-1}(N,G)}, \frac{G}{V_{c-1}(N,G)}\big)\Big|\Big|\frac{[V_{c-1}(S,F)RV^*F]}{[RV^*F]}\Big|;$

(ii) $d\big(\mathcal{WVM}(N,G)\big) \leq d\Big(\mathcal{WVM}\big(\frac{N}{V_{c-1}(N,G)}, \frac{G}{V_{c-1}(N,G)}\big)\Big) + d\big(\frac{[V_{c-1}(S,F)RV^*F]}{[RV^*F]}\big);$

(iii) $e\big(\mathcal{WVM}(N,G)\big)$ *divides*

$$e\Big(\mathcal{WVM}\big(\tfrac{N}{V_{c-1}(N,G)},\tfrac{G}{V_{c-1}(N,G)}\big)\Big)e\big(\tfrac{[V_{c-1}(S,F)RV^*F]}{[RV^*F]}\big).$$

where $e(X)$ *and* $d(X)$ *are the exponent and the minimal number of generators of a group* X*, respectively.*

Proof Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups and G be a group in the variety $\mathcal{W}$ with two normal subgroups K and N such that $K\subseteq N$. Then, by Theorem 2.2 of [12], the following sequence is exact:

$$1\to\mathcal{WVM}(G,K)\overset{\alpha}{\to}\mathcal{WVM}(N,G)$$
$$\to\mathcal{WVM}(N/K,G/K)\overset{\beta}{\to}\frac{K\cap[NV^*G]}{[KV^*G]}\to 1.$$

Thus,

$$|\mathcal{WVM}(N,G)|=|Im(\alpha)|\Big|\frac{R\cap[V_{c-1}(S,F)RV^*F]}{[RV^*F]}\Big|$$

and

$$\frac{\mathcal{WVM}(N/K,G/K)}{Im(\alpha)}\cong K,$$

where $K=V_{c-1}(N,G)$. Hence,

$$|K||\mathcal{WVM}(N,G)|=|\mathcal{WVM}(\frac{N}{K},\frac{G}{K})|\Big|\frac{R\cap[V_{c-1}(S,F)RV^*F]}{[RV^*F]}\Big|.$$

But $[KV^*G]=[V_{c-1}(N,G)V^*G]=V_c(N,G)=\langle e\rangle$, so

$$[V_{c-1}(S,F)RV^*F]\subseteq R.$$

This implies part (i). Similarly, we can prove (ii) and (iii).

Theorem 6 *Let* (N,G) *be a pair of finite groups such that* $V^*(G)\subseteq N$*. Let* $H=G/V^*(G)$ *and* $L=N/V^*(G)$*. Then*

(i) $|[NV^*G]|\leq|\mathcal{WVM}(L,H)|\ |[LV^*H]|\leq|\mathcal{WVM}(N,G)||[NV^*G]|$*;*

(ii) $|[NV^*G]|=|\mathcal{WVM}(L,H)||[LV^*H]|$
$\iff\mathcal{WVM}(L,H)\cong V^*(G)\cap[NV^*G]$*;*

(iii) $|\mathcal{WVM}(L,H)||[LV^*H]|=|\mathcal{WVM}(N,G)||[NV^*G]|$
$\iff\frac{\mathcal{WVM}(L,H)}{\mathcal{WVM}(N,G)}\cong V^*(G)\cap[NV^*G]$.

Proof (i) By Theorem 2.2 of [12], we have

$$|\mathcal{WVM}(L,H)|=|V^*(G)\cap[NV^*G]||\ker\beta|.$$

On the other hand,

$$[LV^*H]=\frac{[NV^*G]V^*(G)}{V^*(G)}\cong\frac{[NV^*G]}{V^*(G)\cap[NV^*G]}.$$

So,

$$|V^*(G)\cap[NV^*G]| = \frac{|[NV^*G]|}{|[LV^*H]|}.$$

Hence,

$$|[NV^*G]||\ker\beta| = |\mathcal{WVM}(L,H)||[LV^*H]|.$$

This implies that

$$|[NV^*G]| \leq |\mathcal{WVM}(L,H)||[LV^*H]|.$$

Moreover, $|\ker\beta| = |Im\alpha| \leq |\mathcal{WVM}(N,G)|$. Therefore,

$$|\mathcal{WVM}(L,H)||[LV^*H]| \leq |\mathcal{WVM}(N,G)||[NV^*G]|.$$

(ii) By considering the first part, we have

$$|\ker\beta| = 1 \Longleftrightarrow \mathcal{WVM}(L,H)\cong V^*(G)\cap[NV^*G] \text{ and}$$
$$|\ker\beta| = 1 \Longleftrightarrow |[NV^*G]| = |\mathcal{WVM}(L,H)||[LV^*H]|.$$

Thus, the result holds.

(iii) By Theorem 2.2 of [12],

$$|\ker\alpha||\mathcal{WVM}(L,H)||[LV^*H]| = |\mathcal{WVM}(N,G)||[NV^*G]|.$$

Also $|\ker\alpha| = 1$ if and only if $\frac{\mathcal{WVM}(L,H)}{\mathcal{WVM}(N,G)} \cong V^*(G)\cap[NV^*G]$, which completes the proof.

By Theorem 6, we obtain the following corollary.

Corollary 2 *Let G be a finite group and $H = G/V^*(G)$. Then*

(i) $|V(G)| \leq |\mathcal{WVM}(H)||V(H)| \leq |\mathcal{WVM}(G)||V(G)|$;
(ii) $|V(G)| = |\mathcal{WVM}(H)||V(H)| \Longleftrightarrow \mathcal{WVM}(H)\cong V^*(G)\cap V(G)$;
(iii) $|\mathcal{WVM}(H)||V(H)| = |\mathcal{WVM}(G)||V(G)| \Longleftrightarrow \frac{\mathcal{WVM}(H)}{\mathcal{WVM}(G)} \cong V^*(G)\cap V(G)$.

3 Isologism of pairs of groups

In this section, we survey some results on isologism of pairs of groups. The notion of isologism of a pair of groups was discussed in [4]. Indeed, we extend some results of [6, 7].

Let (N,G) and (M,H) be pairs of groups. An homomorphism from (N,G) to (M,H) is a homomorphism $f: G\to H$ such that $f(N)\subseteq M$. We say that (N,G) and (M,H) are isomorphic and write $(N,G)\simeq(M,H)$, if f is an isomorphism and $f(N) = M$.

Definition 1 Let (N,G) and (M,H) be two pairs of groups and $\mathcal{V}$ be a variety of groups defined by the set of laws V. An $\mathcal{V}$-isologism between (N,G) and (M,H) is a pair of isomorphism (α,β) with $\alpha : G/V^*(N,G) \to H/V^*(M,H)$ and $\beta : V(N,G) \to V(M,H)$, such that $\alpha\big(N/V^*(N,G)\big) = M/V^*(M,H)$ and for every $v \in V$, $n \in N$ and $g_1,\ldots,g_r \in G$

$$\beta\big(v(g_1,\cdots,g_in,\cdots,g_r)v(g_1,\cdots,g_r)^{-1}\big) = \\ v(h_1,\cdots,h_im,\cdots,h_r)v(h_1,\cdots,h_r)^{-1},$$

whenever, $h_i \in \alpha\big(g_iV^*(N,G)\big)$ and $m \in \alpha\big(nV^*(N,G)\big)$. We say that (N,G) and (M,H) are v-isologic, if there exists an $\mathcal{V}$-isologism between them. In this case we write $(N,G) \sim_{\mathcal{V}} (M,H)$.

If $\mathcal{V}$ is the variety of abelian groups or nilpotent groups of class at most n, then $\mathcal{V}$-isologism coincides with isoclinism and n-isoclinism between pairs of groups. In addition, if $N = G$ and $M = H$, then $\mathcal{V}$-isologism between two pairs of groups is an $\mathcal{V}$-isologism between G and H.
The following Lemma is proved by the authors in ([4], Lemma 5).

Lemma 1 *Let (N,G) be a pair of groups. If M is a normal subgroup of G with $M \leq N$ and H is a subgroup of G, then*

(a) $(H\cap N,H) \sim_{\mathcal{V}} \big((H\cap N)V^*(N,G), HV^*(N,G)\big)$. *In particular if* $G = HV^*(N,G)$, *then* $(H\cap N,H) \sim_{\mathcal{V}} (N,G)$. *Conversely, if* $\frac{H}{V^*(H\cap N,H)}$ *satisfies the ascending chain condition on normal subgroups and* $(H\cap N,H) \sim_{\mathcal{V}} (N,G)$, *then* $G = HV^*(N,G)$;
(b) $(N/M,G/M) \sim_{\mathcal{V}} \big(N/M\cap V(N,G), G/M\cap V(N,G)\big)$. *In particular if* $M\cap V(N,G) = \langle e\rangle$, *then* $(N,G) \sim_{\mathcal{V}} (\frac{N}{M},\frac{G}{M})$. *Conversely, if* $V(N,G)$ *satisfies the ascending chain condition on normal subgroups and* $(N,G) \sim_{\mathcal{V}} (\frac{N}{M},\frac{G}{M})$, *then* $M\cap V(N,G) = \langle e\rangle$.

A pair of groups (N,G) is said to be $\mathcal{V}$-perfect, if $N = V(N,G)$.
The following results give the connections between $\mathcal{V}$-perfect and $\mathcal{V}$-isologism of pairs of groups.

Corollary 3 *Let (N,G) be a $\mathcal{V}$-perfect pair of groups such that $V^*(N,G) = \langle e\rangle$. Then any $\mathcal{V}$-isologic (K,H) to (N,G) is isomorphic to the direct product of N by the marginal subgroup of (K,H).*

Proof By the assumption, we have

$$K/V^*(K,H) \cong N/V^*(N,G) \cong V \sim_{\mathcal{V}} K$$

Now, by Lemma 1, we obtain

$$K = V(K,H)V^*(K,H) \quad \textit{and} \quad V^*(K,H)\cap V(K,H) = \langle e\rangle$$

Thus, $K \cong N \times V^*(K,H)$.

Corollary 4 *Let (N, G) be a pair of finite groups. If H is a normal subgroup of G such that $H \subseteq N$ and the pair (H, N) is $\mathcal{V}$-perfect. Also, let $(H, N) \sim_{\mathcal{V}} (N, G)$, then $N = V(N, G)V^*(N, G)$.*

Proof By Lemma 1, we have $N = HV^*(N, G)$. On the other hand, $H = V(H, N)$. Therefore, $N = V(N, G)V^*(N, G)$.

Corollary 5 *Let (N, G) be a $\mathcal{V}$-perfect pair of groups. Then (N, G) can not be $\mathcal{V}$-isologic to any pair of groups (H, N), in which H is a proper subgroup of G such that $H \subseteq N$ or factor pair of groups of itself.*

Proof If H is a proper subgroup of G such that $H \subseteq N$ and $(N, G) \sim_{\mathcal{V}} (H, N)$, then by using Lemma 1, we have $H \cap V(N, G) = \langle e \rangle$. Therefore, $H = \langle e \rangle$.

Corollary 6 *Let (N, G) and (K, H) be two pairs of groups such that $|N| = |K|$ and $(K, H) \sim_{\mathcal{V}} (N, G)$. If (N, G) is $\mathcal{V}$-perfect or $V^*(N, G) = \langle e \rangle$, then $(N, G) \cong (K, H)$.*

Proof By the definition of isologism, there are isomorphisms

$$\alpha : \frac{N}{V^*(N, G)} \to \frac{K}{V^*(K, H)} \quad \textit{and} \quad \beta : V(N, G) \to V(K, H).$$

Now, if $N = V(N, G)$, then $|N| = |V(K, H)|$ since $|N| = |K|$, it implies that $K = V(K, H)$ and hence, $(N, G) \cong (K, H)$. If $V^*(N, G) = \langle e \rangle$, then the result holds.

Definition 2 Let (N, G) be a pair of groups. If G contains no proper subgroup H satisfying $G = HV^*(N, G)$, then (N, G) is called *subgroup irreducible with respect to* $\mathcal{V}$-isologism. If the group G contains no normal subgroup M with $N \cap M \neq \langle e \rangle$ and $M \cap V(N, G) = \langle e \rangle$, then (N, G) is called *quotient irreducible with respect to* $\mathcal{V}$-isologism.

Lemma 2 *If (N, G) is a $\mathcal{V}$-perfect pair of groups. Then (N, G) is subgroup and quotient irreducible pair of groups.*

Proof Assume that (N, G) be a $\mathcal{V}$-perfect pair of groups and H be a subgroup of G such that $N = HV^*(N, G)$. Thus, we have $V(N, G) = V(H, N)$. So, $H = N$. Now, we can see that (N, G) is quotient irreducible pair of groups.

Theorem 7 *Let (N_1, G_1) and (N_2, G_2) be two $\mathcal{V}$-isologic pairs of groups. If (N_1, G_1) is subgroup and quotient irreducible pair of groups, then so is (N_2, G_2).*

Proof Let H be a normal subgroup of G_1 such that $H \subseteq N_1$ and $H \cap V(N_1, G_1) = \langle e \rangle$. Then, we have

$$H \subseteq V^*(N_1, G_1) \quad \textit{and} \quad V^*\big(N_1/H, G_1/H\big) = \frac{V^*(N_1, G_1)}{H}.$$

Now, assume that

$$N_2 = N_1/H = K/HV^*\big(N_1/H, G_1/H\big) = \frac{K}{H} \cdot \frac{V^*(N_1, G_1)}{H}.$$

So, $N_1 = KV^*(N_1, G_1)$, which implies that $N_1 = K$ and hence $N_2 \cong N_1/H = K/H$. Thus, the result holds when N_1 is assumed to be quotient irreducible pair of groups.

References

1. G. Ellis, The Schur multiplier of a pair of groups, *Appl. Categ. Structures* **6**(3), 355-371 (1998).
2. A. Gholami, Z. Mohammad Abadi and S. Heidarian, Some generalized varietal properties of a pair of groups, *Southeast Asian Bull. Math* **36**, 301-308 (2012).
3. M. Hassanzadeh, A. Pourmirzaei and S. Kayvanfar, On the nilpotency of a pair of groups, *Southeast Asian Bull. Math* **37**, 67-77 (2013).
4. S. Heidarian, A. Gholami and Z. Mohammad Abadi, The structure of ν-isologic pairs of groups, Filomat **26** (1), 67-79 (2011).
5. C.R. Leedham-Green and S. Mackay, *Baer- Invariants, Isologism, Varietal Lows and Homology*, Acta Math **137**, 99-150 (1976).
6. S. Lotfi and M. Taheri, Some relations between isologic and varietal perfect groups, *Hindawi Journal of Mathematics*, Volume, ID5093983, 4pages (2016).
7. M.R.R. Moghaddam and A.R. Salemkar, Varietal isologism and covering groups, *Arch. Math.* **75** (1) 8-15 (2000).
8. M.R.R. Moghaddam, A.R. Salemkar and H.M. Sanny, Some inequalities for the Baer-invariant of a finite groups, *Indag. Math* . **18**(1), 73-82 (2007).
9. M.R.R. Moghaddam, A.R. Salemkar and P. Niroomand, Covering groups respect to two varieties of groups, *Southeast Asian Bull. Math.* **31**, 313-320 (2007).
10. H. Neuman, Varieties of groups, *Speringer- Verlag Berlin Heidelberg, New York,* 1967.
11. A. Onsory and M. Araskhan, The Generalization of the Stalling's Theorem, *Journal of Generalized Lie Theory and Applications*, 3 pages, 2015.
12. M. R. Rismanchian and M. Araskhan, Generalized Baer-Invariant of a pair of groups and the direct limit, *Journal of Sciences, Islamic Republic of Iran*, **22** (2), 159-164 (2011).
13. M. R. Rismanchian and M. Araskhan, Generalized Baer-Invariant of a Pair of Groups and Marginal Extension, *Journal of Sciences, Islamic Republic of Iran*, **23** (3), 251-256 (2012).
14. M. R. Rismanchian and M. Araskhan, Some properties on the Baer-invariant of a pair of groups and ν_G-marginal series, *Turk. J. Math.* **37**, 259-266 (2013).
15. S. M. Taheri and F. Shahini, Varietal nilpotent groups, *World Applied Sciences Journal.* **12**(12), 2314-2316 (2011).