

Perfect 4-Colorings of the 3-Regular Graphs of Order 10

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Received: 26 July 2021 / Accepted: 3 October 2021

Abstract The perfect m -coloring with matrix $A = [a_{ij}]_{i,j \in \{1, \dots, m\}}$ of a graph $G = (V, E)$ with $\{1, \dots, m\}$ color is a vertices coloring of G with m -color so that number of vertex in color j adjacent to a fixed vertex in color i is a_{ij} , independent of the choice of vertex in color i . The matrix $A = [a_{ij}]_{i,j \in \{1, \dots, m\}}$ is called the parameter matrix.

We study the perfect 4-colorings of the 3-regular graphs of order 10, that is, we determine a list of all color parameter matrices corresponding to perfect colorings of 3-regular graphs of order 10.

Keywords Perfect coloring · Parameter matrices · Cubic graph · Equitable partition

Mathematics Subject Classification (2010) 03E02 · 05C15 · 68R05

1 Introduction

The concept of a perfect m -coloring plays an important role in graph theory, algebraic combinatorics, and coding theory (Completely regular codes). There is another term for this concept in the literature as "equitable partition" (see [9]).

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The existence of completely regular codes in graphs is a historical problem in mathematics. Completely regular codes are a generalization of perfect codes. In 1973, Delsarte conjectured the non-existence of perfect codes in Johnson graphs. Therefore, some effort has been done on enumerating the parameter matrices of some Johnson graphs, including $J(6, 3)$, $J(7, 3)$, $J(8, 3)$, $J(8, 4)$, and $J(n, 3)$ (n odd) (see [3,4,8]).

Fon-Der-Flaass enumerated the parameter matrices of n -dimensional hypercube Q_n for $n < 24$. He also obtained some constructions and a necessary condition for the existence of perfect 2-colorings of the n -dimensional hypercube with a given parameter matrix (see [5–7]).

In [2] all perfect 3-colorings of the cubic graphs of order 10 were described and in [10] all Perfect 4-colorings of the 3-regular graphs of order at most 8 were described.

In this paper we enumerate the parameter matrices of all perfect 4-colorings of the 3-regular graphs of order 10.

2 Preliminaries

In this section we use the following definition.

Definition 1 For each graph G and each integer m , a mapping $T : V(G) \rightarrow \{1, \dots, m\}$ is called a perfect m -coloring with matrix $A = [a_{ij}]_{i,j \in \{1, \dots, m\}}$, if it is surjective and for all i, j for every vertex of color i , the number of its neighbors of color j is equal to a_{ij} . The matrix A is called the parameter matrix of a perfect coloring.

The spectrum of a matrix A , denoted by $\sigma(A)$ is the set of all eigenvalues of A . The set of eigenvalues of the adjacency matrix of graph G is called the spectrum of G .

We denoted $M_r(4)$ for all parameter matrices of the perfect 4-colorings of r -regular graphs. Note that if $A \in M_r(4)$, then the total number of entries for each row is equal to r .

If $A = [a_{ij}]_{n \times n}$ is a perfect 4-colorings matrix for a 3-regular graph $G = (V, E)$, then $\sum_{j=1}^4 a_{ij} = 3$ for all $1 \leq i \leq 4$. So there are 20 different models for each row of matrices. Hence there are 20^4 matrices.

Let $A = [a_{ij}]_{4 \times 4}$ be a 4-color parameter matrix for a graph $G = (V, E)$. The first observation says A must possess a weak form of symmetry, described in the following lemma:

Lemma 1 Suppose $A = [a_{ij}]_{n \times n}$ is a parameter matrix for a graph $G = (V, E)$. Then, $a_{ij} = 0$ if and only if $a_{ji} = 0$ for $1 \leq i, j \leq n$.

Definition 2 Let A and B are two parameter matrices of the perfect 4-colorings of graph G . We define A and B are equivalent if A transformed to B by a permutation on colors.

We have the obvious lemmas:

Lemma 2 Let $A = [a_{ij}]_{4 \times 4}$ and $A \in M_3(4)$ and $\sigma \in S_4$ (where S_4 is the symmetric group of degree 4). Then $[a_{ij}]_{4 \times 4} \sim [a_{i\sigma(j)}]_{4 \times 4}$.

Lemma 3 Let $A = [a_{ij}]_{4 \times 4} \in M_3(4)$. Then the following cases do not happen:

- 1) $a_{14} = 0, a_{13} = 0, a_{12} = 0$;
- 2) $a_{24} = 0, a_{23} = 0, a_{21} = 0$;
- 3) $a_{34} = 0, a_{32} = 0, a_{31} = 0$;
- 4) $a_{43} = 0, a_{42} = 0, a_{41} = 0$.

Lemma 4 Suppose $A \in M_3(4)$. Then there is not $\sigma \in S_4$ such that $[a_{i\sigma(j)}] =$

$$\begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}$$

Proof It is clear with connectivity.

Remark 1 Suppose $A \in M_3(4)$ is a parameter matrix for a 3-regular graph G .

If there is $\sigma \in S_4$ such that $A = [a_{i\sigma(j)}] = \begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{bmatrix}$, then G is bipartite.

To see this V is the set of vertices of G . Divided V in to two independent sets V_1 and V_2 with color numbers 3, 4 and 1, 2 respective. Therefore G is a bipartite graph.

It is easy to see that each perfect coloring on a graph G , create an equitable partition. So, we have the following lemma.

Lemma 5 Suppose $A \in M_3(4)$ is a coloring matrix for graph G . Then the spectrum of A is a subset of the spectrum of G .

Lemma 6 If $A \in M_3(4)$, then all of the eigenvalues of A are real.

Proof By symmetry of adjacency matrices of G is obvious.

Proposition 1 Let $A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$ be a color incidence matrix of some

connected graph $G = (V, E)$, and suppose that $|v|$ denote the number of vertices of G and v_i denote color i ; ($1 \leq i \leq 4$).

1) If $b \neq 0, c \neq 0$ and $d \neq 0$, then

$$v_1 = \frac{|v|}{1 + \frac{b}{e} + \frac{c}{i} + \frac{d}{m}}, v_2 = \frac{|v|}{\frac{e}{b} + 1 + \frac{ec}{bi} + \frac{ed}{bm}}$$

$$v_3 = \frac{|v|}{\frac{i}{c} + \frac{ib}{ce} + 1 + \frac{id}{cm}}, v_4 = \frac{|v|}{\frac{m}{d} + \frac{mb}{de} + \frac{mc}{di} + 1}.$$

2) If $b \neq 0$, $c \neq 0$ and $h \neq 0$, then

$$v_1 = \frac{|v|}{1 + \frac{b}{e} + \frac{c}{i} + \frac{bh}{en}}, v_2 = \frac{|v|}{\frac{e}{b} + 1 + \frac{ec}{bi} + \frac{h}{n}},$$

$$v_3 = \frac{|v|}{\frac{i}{c} + \frac{ib}{ce} + 1 + \frac{ibh}{cen}}, v_4 = \frac{|v|}{\frac{ne}{hb} + \frac{n}{h} + \frac{nec}{hbi} + 1}.$$

3) If $b \neq 0$, $c \neq 0$ and $l \neq 0$, then

$$v_1 = \frac{|v|}{1 + \frac{b}{e} + \frac{c}{i} + \frac{cl}{io}}, v_2 = \frac{|v|}{\frac{e}{b} + 1 + \frac{ec}{bi} + \frac{ecl}{bio}},$$

$$v_3 = \frac{|v|}{\frac{i}{c} + \frac{ib}{ce} + 1 + \frac{l}{o}}, v_4 = \frac{|v|}{\frac{oi}{lc} + \frac{oib}{lce} + \frac{o}{l} + 1}.$$

4) If $b \neq 0$, $d \neq 0$ and $g \neq 0$, then

$$v_1 = \frac{|v|}{1 + \frac{b}{e} + \frac{bg}{ej} + \frac{d}{m}}, v_2 = \frac{|v|}{\frac{e}{b} + 1 + \frac{g}{j} + \frac{ed}{bm}},$$

$$v_3 = \frac{|v|}{\frac{je}{gb} + \frac{j}{g} + 1 + \frac{jed}{gbm}}, v_4 = \frac{|v|}{\frac{m}{d} + \frac{mb}{de} + \frac{mbg}{dej} + 1}.$$

5) If $b \neq 0$, $d \neq 0$ and $l \neq 0$, then

$$v_1 = \frac{|v|}{1 + \frac{b}{e} + \frac{do}{ml} + \frac{d}{m}}, v_2 = \frac{|v|}{\frac{e}{b} + 1 + \frac{edo}{bml} + \frac{ed}{bm}},$$

$$v_3 = \frac{|v|}{\frac{lm}{od} + \frac{lmb}{ode} + 1 + \frac{l}{o}}, v_4 = \frac{|v|}{\frac{m}{d} + \frac{mb}{de} + \frac{o}{l} + 1}.$$

6) If $b \neq 0$, $g \neq 0$ and $h \neq 0$, then

$$v_1 = \frac{|v|}{1 + \frac{b}{e} + \frac{bg}{ej} + \frac{bh}{en}}, v_2 = \frac{|v|}{\frac{e}{b} + 1 + \frac{g}{j} + \frac{h}{n}},$$

$$v_3 = \frac{|v|}{\frac{je}{gb} + \frac{j}{g} + 1 + \frac{jh}{gn}}, v_4 = \frac{|v|}{\frac{ne}{hb} + \frac{n}{h} + \frac{ng}{hj} + 1}.$$

7) If $b \neq 0$, $g \neq 0$ and $l \neq 0$, then

$$v_1 = \frac{|v|}{1 + \frac{b}{e} + \frac{bg}{ej} + \frac{bgl}{ejo}}, v_2 = \frac{|v|}{\frac{e}{b} + 1 + \frac{g}{j} + \frac{gl}{jo}},$$

$$v_3 = \frac{|v|}{\frac{je}{gb} + \frac{j}{g} + 1 + \frac{l}{o}}, v_4 = \frac{|v|}{\frac{oje}{lgb} + \frac{oj}{lg} + \frac{o}{l} + 1}.$$

8) If $b \neq 0$, $h \neq 0$ and $l \neq 0$, then

$$v_1 = \frac{|v|}{1 + \frac{b}{e} + \frac{bho}{enl} + \frac{bh}{en}}, v_2 = \frac{|v|}{\frac{e}{b} + 1 + \frac{ho}{nl} + \frac{h}{n}},$$

$$v_3 = \frac{|v|}{\frac{lne}{ohb} + \frac{ln}{oh} + 1 + \frac{l}{o}}, v_4 = \frac{|v|}{\frac{ne}{hb} + \frac{n}{h} + \frac{o}{l} + 1}.$$

9) If $c \neq 0$, $d \neq 0$ and $g \neq 0$, then

$$v_1 = \frac{|v|}{1 + \frac{cj}{ig} + \frac{c}{i} + \frac{d}{m}}, v_2 = \frac{|v|}{\frac{gi}{cj} + 1 + \frac{g}{j} + \frac{gid}{jcm}},$$

$$v_3 = \frac{|v|}{\frac{i}{c} + \frac{j}{g} + 1 + \frac{id}{cm}}, v_4 = \frac{|v|}{\frac{m}{d} + \frac{mcj}{dig} + \frac{mc}{di} + 1}.$$

10) If $c \neq 0$, $d \neq 0$ and $h \neq 0$, then

$$v_1 = \frac{|v|}{1 + \frac{dn}{mh} + \frac{c}{i} + \frac{d}{m}}, v_2 = \frac{|v|}{\frac{hm}{dn} + 1 + \frac{hmc}{ndi} + \frac{h}{n}},$$

$$v_3 = \frac{|v|}{\frac{i}{c} + \frac{idn}{cmh} + 1 + \frac{id}{cm}}, v_4 = \frac{|v|}{\frac{m}{d} + \frac{n}{h} + \frac{mc}{di} + 1}.$$

11) If $c \neq 0$, $g \neq 0$ and $h \neq 0$, then

$$v_1 = \frac{|v|}{1 + \frac{cj}{ig} + \frac{c}{i} + \frac{cjh}{ign}}, v_2 = \frac{|v|}{\frac{gi}{jc} + 1 + \frac{g}{j} + \frac{h}{n}},$$

$$v_3 = \frac{|v|}{\frac{i}{c} + \frac{j}{g} + 1 + \frac{jh}{gn}}, v_4 = \frac{|v|}{\frac{ngi}{hjc} + \frac{n}{h} + \frac{ng}{hj} + 1}.$$

12) If $c \neq 0$, $g \neq 0$ and $l \neq 0$, then

$$v_1 = \frac{|v|}{1 + \frac{cj}{ig} + \frac{c}{i} + \frac{cl}{io}}, v_2 = \frac{|v|}{\frac{gi}{jc} + 1 + \frac{g}{j} + \frac{gl}{jo}},$$

$$v_3 = \frac{|v|}{\frac{i}{c} + \frac{j}{g} + 1 + \frac{l}{o}}, v_4 = \frac{|v|}{\frac{oi}{lc} + \frac{oj}{lg} + \frac{o}{l} + 1}.$$

13) If $c \neq 0$, $h \neq 0$ and $l \neq 0$, then

$$v_1 = \frac{|v|}{1 + \frac{cln}{ioh} + \frac{c}{i} + \frac{cl}{io}}, v_2 = \frac{|v|}{\frac{hoi}{nlc} + 1 + \frac{ho}{nl} + \frac{h}{n}},$$

$$v_3 = \frac{|v|}{\frac{i}{c} + \frac{ln}{oh} + 1 + \frac{l}{o}}, v_4 = \frac{|v|}{\frac{oi}{lc} + \frac{n}{h} + \frac{o}{l} + 1}.$$

14) If $d \neq 0$, $g \neq 0$ and $h \neq 0$, then

$$v_1 = \frac{|v|}{1 + \frac{dn}{mh} + \frac{dng}{mhj} + \frac{d}{m}}, v_2 = \frac{|v|}{\frac{hm}{nd} + 1 + \frac{g}{j} + \frac{h}{n}},$$

$$v_3 = \frac{|v|}{\frac{jhm}{gnd} + \frac{j}{g} + 1 + \frac{jh}{gn}}, v_4 = \frac{|v|}{\frac{m}{d} + \frac{n}{h} + \frac{ng}{hj} + 1}.$$

15) If $d \neq 0$, $g \neq 0$ and $l \neq 0$, then

$$v_1 = \frac{|v|}{1 + \frac{doj}{mlg} + \frac{do}{ml} + \frac{d}{m}}, v_2 = \frac{|v|}{\frac{glm}{jod} + 1 + \frac{g}{j} + \frac{gl}{jo}},$$

$$v_3 = \frac{|v|}{\frac{lm}{od} + \frac{j}{g} + 1 + \frac{l}{o}}, v_4 = \frac{|v|}{\frac{m}{d} + \frac{oj}{lg} + \frac{o}{l} + 1}.$$

16) If $d \neq 0$, $h \neq 0$ and $l \neq 0$, then

$$v_1 = \frac{|v|}{1 + \frac{dn}{mh} + \frac{do}{ml} + \frac{d}{m}}, v_2 = \frac{|v|}{\frac{hm}{nd} + 1 + \frac{ho}{nl} + \frac{h}{n}},$$

$$v_3 = \frac{|v|}{\frac{lm}{od} + \frac{ln}{oh} + 1 + \frac{l}{o}}, v_4 = \frac{|v|}{\frac{m}{d} + \frac{n}{h} + \frac{o}{l} + 1}.$$

By using above proposition and lemmas, for $n=10$ we only have the following matrices, which we have shown with M_1, \dots, M_{28} .

$$M_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}, M_3 = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, M_4 = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 3 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

$$M_5 = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, M_6 = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \\ 3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}, M_7 = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, M_8 = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix},$$

$$M_9 = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, M_{10} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, M_{11} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}, M_{12} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

$$M_{13} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, M_{14} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}, M_{15} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, M_{16} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 \end{bmatrix},$$

$$M_{17} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{bmatrix}, M_{18} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}, M_{19} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, M_{20} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

$$M_{21} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, M_{22} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}, M_{23} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}, M_{24} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix},$$

$$M_{25} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}, M_{26} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, M_{27} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, M_{28} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix},$$

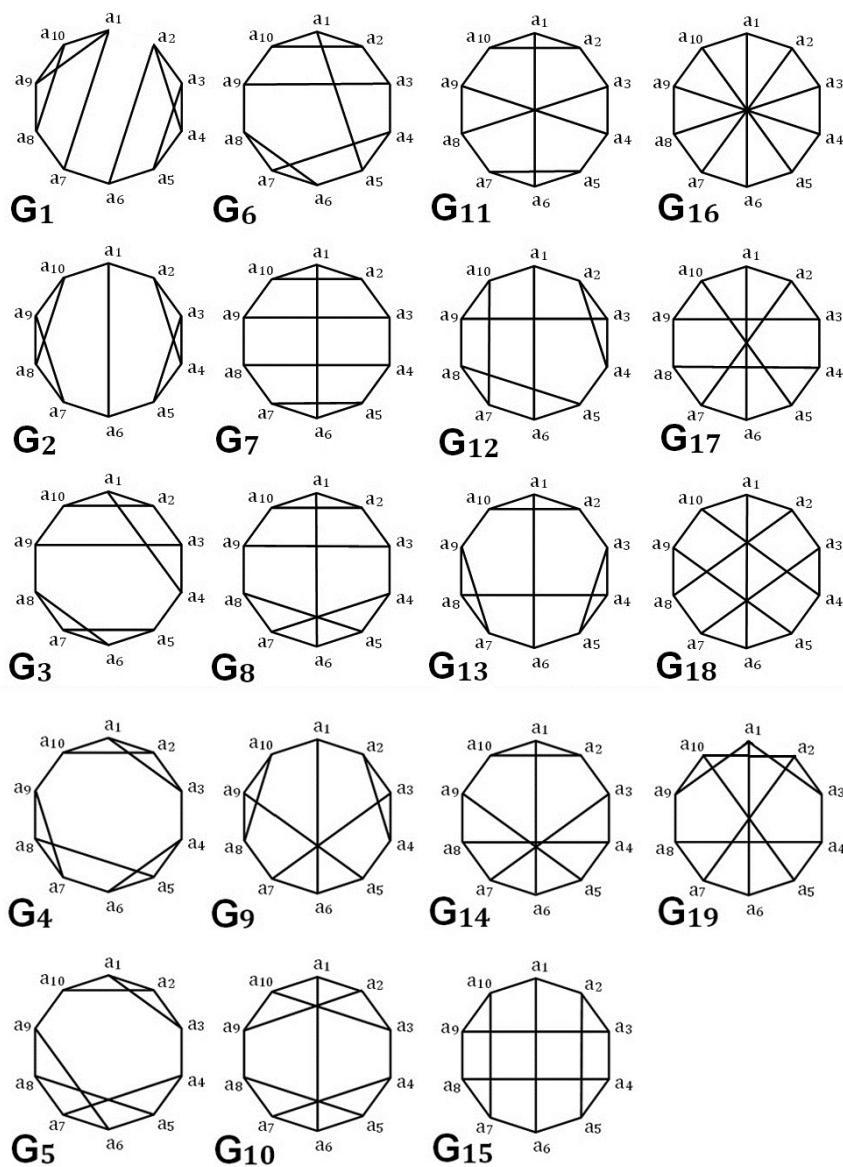


Fig. 1: Connected 3-regular graphs of order 10

3 Main Results

A cubic graph is a 3-regular graph and there are 19 none isomorphic cubic graphs of order 10 as shown below in Fig. 1.

Table 1

Graph	Matrix											
G_1	M_1	M_7	M_{16}	M_{24}	M_{26}							
G_2	M_1	M_2	M_7	M_9	M_{16}	M_{24}	M_{26}					
G_3	M_{15}											
G_4	M_{23}	M_{26}										
G_5	M_{23}	M_{26}										
G_6	M_1	M_7	M_{16}	M_{24}	M_{26}							
G_8	M_5											
G_9	M_2	M_4	M_7	M_8	M_9	M_{16}	M_{20}					
G_{10}	M_1	M_2	M_4	M_6	M_7	M_8	M_{15}	M_{20}	M_{24}	M_{26}	M_{28}	
G_{12}	M_{15}	M_{18}										
G_{13}	M_9											
G_{15}	M_{19}											
G_{17}	M_{10}	M_{19}										
G_{18}	M_1	M_2	M_7	M_9	M_{16}	M_{24}	M_{26}					
G_{19}	M_2	M_9										

Note: By using Lemma 6 there are no perfect 4-colorings with the matrices M_1, \dots, M_{28} for graphs $G_7, G_{11}, G_{14}, G_{16}$, and M_1, \dots, M_{28} can be parameter matrices of order 10 for graphs $G_1, G_2, G_3, G_4, G_5, G_6, G_8, G_9, G_{10}, G_{12}, G_{13}, G_{15}, G_{17}, G_{18}, G_{19}$ are listed in Table 1.

Theorem 1 *The parameter matrices of 3-regular graphs of order 10 are listed in the Table 2. (Checkmark(✓) means having perfect 4-colorings and cross(×) means not having perfect 4-colorings.)*

Table 2: The parameter matrices of 3-regular graphs of order 10

Graph	Matrix																	
	1	2	4	5	6	7	8	9	10	15	16	18	19	20	23	24	26	28
G_1	×					×					×					×	×	
G_2	×	×				×		×			×					×	×	
G_3										×								
G_4															×		×	
G_5															×		×	
G_6	×					×					✓					✓	×	
G_8				×														
G_9		×	✓			×	×	×			✓			×				
G_{10}	×	×	×		×	×	×			×				×		×	×	×
G_{12}										×		×						
G_{13}								✓										
G_{15}													×					
G_{17}									×				×					
G_{18}	×	×				×		×			×					✓	×	
G_{19}		✓						✓										

Proof With consideration of 3-regular graphs eigenvalues and using Proposition 1, and Lemmas 5 and 6, it can be seen that the connected 3-regular graphs

with 10 vertices can have perfect 4-colorings with matrices $M_1, M_2, M_4, M_5, M_6, M_7, M_8, M_9, M_{10}, M_{15}, M_{16}, M_{18}, M_{19}, M_{20}, M_{23}, M_{24}, M_{26}$ and M_{28} which are represented by Table 2.

Now we introduce the mappings of all graphs that have perfect 4-colorings with the parameter matrices. The graph G_6 has perfect 4-colorings with the matrices M_{16} and M_{24} .

Consider two mappings T_1 and T_2 as follows :

$$T_1(a_1) = T_1(a_8) = 1, T_1(a_3) = T_1(a_4) = 2,$$

$$T_1(a_2) = T_1(a_5) = T_1(a_7) = T_1(a_9) = 3, T_1(a_6) = T_1(a_{10}) = 4.$$

$$T_2(a_5) = T_2(a_9) = 1, T_2(a_2) = T_2(a_7) = 2,$$

$$T_2(a_1) = T_2(a_6) = T_2(a_8) = T_2(a_{10}) = 3, T_2(a_3) = T_2(a_4) = 4.$$

Can be seen that T_1 and T_2 are perfect 4-colorings with the matrices M_{16} and M_{24} , respectively.

The graph G_9 has perfect 4-colorings with the matrices M_4 and M_{16} . Consider two mappings T_1 and T_2 as follows:

$$T_1(a_8) = T_1(a_9) = T_1(a_{10}) = 1, T_1(a_2) = T_1(a_3) = T_1(a_4) = 2,$$

$$T_1(a_6) = 3, T_1(a_1) = T_1(a_5) = T_1(a_7) = 4.$$

$$T_2(a_2) = T_2(a_9) = 1, T_2(a_6) = T_2(a_7) = 2,$$

$$T_2(a_1) = T_2(a_3) = T_2(a_5) = T_2(a_8) = 3, T_2(a_4) = T_2(a_{10}) = 4.$$

Can be seen that T_1 and T_2 are perfect 4-colorings with the matrices M_4 and M_{16} , respectively.

The graph G_{13} has perfect 4-colorings with the matrix M_9 . Consider the mapping T as follows:

$$T(a_1) = T(a_5) = T(a_7) = 4, T(a_2) = T(a_4) = T(a_9) = 1,$$

$$T(a_3) = T(a_8) = T(a_{10}) = 3, T(a_6) = 2.$$

Can be seen that T is a perfect 4-colorings with the matrix M_9 .

The graph G_{18} has perfect 4-colorings with the matrix M_{24} . Consider the mapping T as follows:

$$T(a_2) = T(a_5) = 1, T(a_7) = T(a_{10}) = 2,$$

$$T(a_3) = T(a_4) = T(a_8) = T(a_9) = 3, T(a_1) = T(a_6) = 4.$$

Can be seen that T is a perfect 4-colorings with the matrix M_{24} .

The graph G_{19} has perfect 4-colorings with the matrices M_2 and M_9 . Consider two mappings T_1 and T_2 as follows:

$$T_1(a_2) = T_1(a_7) = 1, T_1(a_1) = T_1(a_9) = 2,$$

$$T_1(a_4) = T_1(a_5) = 3, T_1(a_3) = T_1(a_6) = T_1(a_8) = T_1(a_{10}) = 4.$$

$$T_2(a_1) = T_2(a_8) = T_2(a_{10}) = 4, T_2(a_9) = 2,$$

$$T_2(a_2) = T_2(a_4) = T_2(a_6) = 3, T_2(a_3) = T_2(a_5) = T_2(a_7) = 1.$$

Can be seen that T_1 and T_2 are perfect 4-colorings with the matrices M_2 and M_9 , respectively.

Here, we prove that other cases are not feasible. Some examples are as follows. The rest of the graphs in Table 2 will be demonstrated in the some order.

We show that the graph G_1 has no perfect 4-colorings with the matrices M_1, M_7, M_{16}, M_{24} and M_{26} . For example we claim that M_{26} has no perfect 4-colorings for graph G_1 . Contrary to our claim, suppose that T is a perfect

4-colorings with the matrix M_{26} for graph G_1 . Then according to the matrix M_{26} , by symmetry we have five cases for the color of number 1 as follows:

1. If $T(a_1) = 1$, because $m_{33} = 0$, thus none two vertices with color 3 shouldn't be adjacent. Therefore $T(a_5) = 3$, $T(a_2) = 3$ or $T(a_3) = 3$. If $T(a_2) = 3$ then $T(a_3) = 4$ therefore $T(a_4) = 2$ so $T(a_6) = 4$ but $T(a_7) = 1$ or $T(a_{10}) = 1$, which is a contradiction because we must colored two vertices adjacent with color 3. If $T(a_3) = 3$, $T(a_2) = 4$, $T(a_4) = 2$ so $T(a_6) = 4$, which is a contradiction with above.
2. $T(a_2) = 1$, it follows that $T(a_1) = T(a_4) = 3$ then $T(a_3) = 4$ because $m_{43} = 1$, which is a contradiction with the fourth row of the matrix M_{26} .
3. If $T(a_3) = 1$, it follows that $T(a_1) = T(a_4) = 3$ so $T(a_2) = 4$ because $m_{43} = 1$, which is a contradiction.
4. If $T(a_4) = 1$ then $T(a_3) = T(a_5) = 3$. It follows that $T(a_2) = 4$ then $T(a_1) = 2$ and $T(a_6) = 4$, which is a contradiction with the first case.
5. If $T(a_5) = 1$ then we have 2 cases;
 $T(a_1) = T(a_4) = 3$, it follows that $T(a_6) = 4$ therefore $T(a_4) = 3$ so $T(a_3) = 2$ and $T(a_2) = 4$ or $T(a_4) = 4$ and $T(a_2) = 3$, in both cases because $m_{43} = 1$; which is a contradiction. Therefore the graph G_1 has no perfect 4-colorings with matrix M_{26} .

Now we show that the graph G_{15} has no perfect 4-colorings with the matrix M_{19} . Contrary to our claim, suppose that T is a perfect 4-colorings with matrix M_{19} for the graph G_{15} . According to the matrix M_{19} , by symmetry we have three cases for the color of number 3 as follows:

1. If $T(a_1) = 3$, according to the matrix M_{19} , there are two situations;
 $T(a_2) = T(a_6) = 1$ or $T(a_2) = T(a_{10}) = 1$. For first case if $T(a_2) = T(a_6) = 1$, then $T(a_{10}) = 2$ therefore $T(a_7) = T(a_9) = 4$ then $T(a_8) = 2$. It follows that $T(a_4) = 3$ and $T(a_5) = 3$, which is a contradiction with the $m_{33} = 0$ and for the second case, if $T(a_2) = T(a_{10}) = 1$ then $T(a_6) = 2$. It follows that $T(a_5) = T(a_7) = 4$ because $T(a_2) = 1$ so $T(a_3) = 3$, similarly $T(a_9) = 3$, which is a contradiction with the $m_{33} = 0$ of the matrix M_{19} .
2. If $T(a_2) = 3$, according to the matrix M_{19} , there are three situations;
 $T(a_1) = T(a_3) = 1$ or $T(a_1) = T(a_5) = 1$ or $T(a_3) = T(a_5) = 1$. For first case if $T(a_1) = T(a_3) = 1$, then $T(a_5) = 2$ therefore $T(a_4) = T(a_8) = 4$, which is a contradiction with the $m_{44} = 1$ of the matrix M_{19} .
For the second case, if $T(a_1) = T(a_5) = 1$ then $T(a_3) = 2$ therefore $T(a_4) = 4$ and $T(a_9) = 4$. It follows that $T(a_8) = 4$, which is a contradiction with the $m_{44} = 1$ of the matrix M_{19} .
For the third case, if $T(a_3) = T(a_5) = 1$, then $T(a_1) = 2$ therefore $T(a_6) = T(a_{10}) = 4$. It follows that $T(a_7) = 4$, which is a contradiction with the $m_{44} = 1$ of the matrix M_{19} .
3. If $T(a_3) = 3$, according to the matrix M_{19} , there are three situations:
 $T(a_2) = T(a_4) = 1$ or $T(a_2) = T(a_9) = 1$ or $T(a_4) = T(a_9) = 1$.
For first case if $T(a_2) = T(a_4) = 1$, then $T(a_9) = 2$ therefore $T(a_8) = T(a_{10}) = 4$. It follows that $T(a_7) = 4$, which is a contradiction with the fourth row of the matrix M_{19} .

For the second case if $T(a_2) = T(a_9) = 1$, then $T(a_4) = 2$ therefore, $T(a_5) = T(a_8) = 4$, on the otherhand $T(a_7) = 4$. It follows that $T(a_6) = 4$, which is a contradiction as above.

For the third case, if $T(a_4) = T(a_9) = 1$, then $T(a_2) = 2$, therefore $T(a_5) = T(a_1) = 4$. It follows that $T(a_6) = 4$, which is a contradiction as above. Therefore the graph G_{15} has no perfect 4-colorings with matrix M_{19} . As it is stated, the graph G_{15} has no perfect 4-colorings with matrix M_{19} .

About the other graphs of order 10, similarly we can get the same result as in Table 2 in Theorem 1.

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