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Numerical Solutions for Fractional Black-Scholes Option Pricing Equation

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Abstract In this article we have applied a numerical finite difference method to solve the Black-Scholes European and American option pricing both presented by fractional differential equations in time and asset.

Keywords Fractional Black-Schole · Numerical solutions · Finite difference.

Mathematics Subject Classification (2010) 26A33 · 65L12

1 Introduction

In the last decades, differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in physics, engineering, economics, finance. For instance, see [4, 5] and references therein.
In early 1970, Black and Scholes introduced the famous model for option pricing. They studied the behavior of asset price and derived a partial differential equation that describes the option value. For this problem in the form of fractional order derivative, the closed form solution is too rare. For American options in both classical and fractional order derivatives the closed form solutions may not be so easy to obtain [6]. Therefore, we may use appropriate numerical methods to solve the problem.

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2 Preliminaries and notations

In this section, we give some definitions and lemmas which are used further in this article.

Definition 1 The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of function $f \in L^1(\mathbb{R}^+)$ is defined as

$$I^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

where $\Gamma(.)$ is the Euler gamma function.

Definition 2 The Riemann-Liouville fractional derivative of order $\alpha > 0$ denoted by D^{α} and defined by

$$D^{\alpha} f(x) = \frac{d^m}{dt^m}(I^{m-\alpha} f(x)),$$

where $m - 1 < \alpha \leq m, m \in \mathbb{N}$ and m is the smallest integer greater than α.

3 Finite Difference Methods

The original classical order Black-Scholes model for the value of an option is described as follows

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + r(t) S \frac{\partial V}{\partial S} - r(t) V = 0, \qquad (S,t) \in \mathbb{R}^+ \times (0,T), \quad (1)$$

where $V(S,t)$ is the European or American put option at asset price S and at time t, T is the maturity, $r(t)$ is the risk free interest rate and σ represents the volatility function of underlying asset. The simplest types of options come in two main brands, Calls and Puts. In particular, a call option allows its owner to buy and a put option to sell its underlying asset, at a certain time t for a fixed strike price K. Therefore, the payoff functions are

$$V_c(S,t) = \max\{S-K, 0\}, \qquad V_p(S,t) = \max\{K-S, 0\},$$

where $V_c(S,t)$ and $V_p(S,t)$ are the value of the call and put options, respectively. In the last decade, many authors studied existing solutions for Black-Scholes model (1) in classical form [1,2]. In this article, we consider the following time and asset fractional Black-Schole equation

$$\frac{\partial^{\alpha} V}{\partial t^{\alpha}} + \frac{\sigma^2 S^2}{2}\frac{\partial^{\beta} V}{\partial S^{\beta}} + rS\frac{\partial V}{\partial S} - rV = 0, \qquad 1 < \beta \leq 2, \ \ 0 < \alpha \leq 1, \quad (2)$$

where $\frac{\partial^\alpha V}{\partial t^\alpha}$ is the Caputo derivative and $\frac{\partial^\beta V}{\partial S^\beta}$ is the Riemann-Liouville derivative. The boundary conditions for the European Put are:

$$\begin{aligned} &V(S,T) = f(S) = \max\{K-S,0\}, \\ &V(0,t) = Ke^{-r(T-t)}, \\ &V(S,0) \to 0 \quad as \quad S \to \infty. \end{aligned} \tag{3}$$

Pricing European put option (2) and boundary condition (3) can be solved by some method such as, Binomial Tree Method, finite difference, Monte-Carlo Simulation, etc. Here, we have used finite difference method to solve both European and American options models presented by FDE. Applying this method to (2), we need initial conditions. To provide this initial condition, we change time variable t by $\tau = T - t$. In this case, equation (2) will transfer into the following equation.

$$\tau^{\alpha-1}(T-\tau)^{\alpha-1}\frac{\partial^\alpha V}{\partial \tau^\alpha} - \frac{\sigma^2 S^2}{2}\frac{\partial^\beta V}{\partial S^\beta} - rS\frac{\partial V}{\partial S} + rV = 0. \tag{4}$$

Now, we discretize equation (4) in the new domain by uniform grid (S_m, τ_n) with $S_m = mh$ $(m = 0, 1, \cdots, M)$ and $\tau_n = nk$ $(n = 0, 1, \cdots, N)$, where $h = S_{\max}/M$, $K = T/N$ and $S_{\max}$ is a maximum value for S, which is sufficiently large. Customary, the finite difference approximation for derivatives [3,7] are

$$\begin{aligned} \frac{\partial^\alpha V}{\partial \tau^\alpha} &= \frac{k^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{n}(V_m^{n-j+1} - V_m^{n-j})(j^{1-\alpha} - (j-1)^{1-\alpha}) + O(k), \\ \frac{\partial V}{\partial S} &= \frac{V_m^n - V_{m-1}^n}{h} + O(h), \\ \frac{\partial^\beta V}{\partial S^\beta} &= \frac{1}{h^\beta}\sum_{j=1}^{m+1} w_j^\beta V_{m-j+1}^n + O(k+h), \end{aligned}$$

where $w_0^\beta = 1, w_1^\beta = -\beta$ and $w_j^\beta = (-1)^j \frac{\beta(\beta-1)\cdots(\beta-j+1)}{j!}$. Substituting these approximations in (4) dicsretized system will be

$$\begin{aligned} (nk)^{\alpha-1}(T-nk)^{1-\alpha}\frac{k^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{n}(V_m^{n-j+1} - V_m^{n-j})(j^{1-\alpha} - (j-1)^{1-\alpha}) \\ - \frac{\sigma^2(mh)^2}{2h^\beta}\sum_{j=1}^{m+1} w_j^\beta V_{m-j+1}^n - rmh\frac{V_m^n - V_{m-1}^n}{h} + rV_m^n = 0. \end{aligned}$$

Denoting

$$\begin{aligned} \gamma_{n,k} &= \frac{(nk)^{\alpha-1}(T-nk)^{1-\alpha}k^\alpha}{\Gamma(2-\alpha)k^\alpha}, \\ \sigma_j &= j^{1-\alpha} - (j-1)^{1-\alpha}, \\ \lambda_{m,h} &= \frac{\sigma^2(mh)^2}{2h^\beta}, \end{aligned}$$

We can rewrite this equation as follows:
if $n = 1$,

$$-\lambda_{m,h}V_{m+1}^1 + (\gamma_{1,k}\sigma_1 - \lambda_{m,h}w_1^\beta - r(m-1))V_m^1 - \lambda_{m,h}\sum_{j=3}^{m+1} w_j^\beta V_{m-j+1}^1 + (rm - \lambda_{m,h}w_2^\beta)V_{m-1}^1 = \gamma_{1,k}\sigma_1 V_m^0, \quad (5)$$

if $n > 1$,

$$\begin{aligned}-\lambda_{m,h}V_{m+1}^n &+ (\gamma_{n,k}\sigma_1 - \lambda_{m,h}w_1^\beta - r(m-1))V_m^n + (rm - \lambda_{m,h}w_2^\beta)V_{m-1}^n \\ &- \lambda_{m,h}\sum_{j=3}^{m+1} w_j^\beta V_{m-j+1}^n = \gamma_{n,k}\sigma_1 V_m^{n-1} - \gamma_{n,k}\sum_{j=2}^{n}\sigma_j(V_m^{n-j+1} - V_m^{n-j}) \\ &= \gamma_{n,k}\sigma_n V_m^0 + \gamma_{n,k}\sum_{j=1}^{n-1}\sigma_j V_m^{n-j}(2j^{1-\alpha} - (j+1)^{1-\alpha} - (j-1)^{1-\alpha}).\end{aligned} \quad (6)$$

We can summarize equations (5) and (6) as following matrix form:

$$\begin{cases} AV^1 = \gamma_{1,k}\sigma_1 V^0 + B_1, \\ AV^{j+1} = \gamma_{j,k}\sigma_j V^0 + \gamma_{j,k}(p_1V^j + p_2V^{j-1} + \cdots + p_jV^1) + B_j, \\ V^0 = f. \end{cases} \quad (7)$$

where $j > 0$ and $V^j = (V_1^j, V_2^j, \cdots, V_{m-1}^j)^T$, $B_j = V_0^j((r+\lambda_{1,h})w_2, \lambda_{2,h}w_3, \cdots, \lambda_{m-1,h}w_m,)^T$, $f = (f(S_1), f(S_2), \cdots, f(S_{m-1}),)^T$, $p_j = 2j^{1-\alpha} - (j+1)^{1-\alpha} - (j-1)^{1-\alpha}$ and matrix $A = (A_{i,j})_{(m-1)\times(m-1)}$ is defined as

$$A = \begin{cases} 0 & j > i+1, \\ -\lambda_{i,h} & j + i + 1, \\ \gamma_{n,k}\sigma_1 - \lambda_{i,h}w_1^\beta - (i-1)r & j = i, \\ ir - \lambda_{i,h}w_2^\beta & j = i+1, \\ -\lambda_{i,h}w_{i-j+1}^\beta & j < i+1, \end{cases} \quad (8)$$

Now, the solution of (2) can be obtained by solving (7). For European option we can invert matrix A to solve the linear system at each time step. In what follows, we present some numerical examples to show the accuracy of our proposed method.

Example 1 Consider (2) and the condition (3) with following parameters

$$\begin{aligned} K = 50, \quad T = 3, \quad r = 0.05, \quad \sigma = 0.2, \\ M = 80, \quad N = 200, \quad S_{\max} = 150. \end{aligned} \quad (9)$$

Implementing our method in a MATLAB code, the results are illustrated in Figure 1 for fractional European option with different values of α and β.

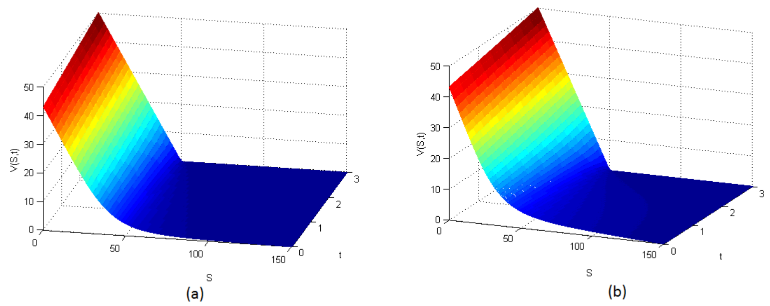

Fig. 1 European Put values calculated using the finite difference method for $\alpha = 0.9$ and (a) $\beta = 1.9$, (b) $\beta = 1.7$

4 American Option

The early exercise property of American option can not be solved with traditional finite difference method. In other words, we can't solve system (7) by inverting matrix A. Therefore, finding the early exercise boundary prior to discretization on underlying asset is necessary in each time step. i.e., we need to check the possibility of early exercise in an explicit scheme

$$V_m^n = \max(V_m^n, K - mh).$$

However, this is difficult to do with an implicit scheme as computing V_m^n requires knowing the other V_m^n. To resolve this difficulty, We can use an iterative method to solve the linear system. Here, we use *successive over relaxation* or SOR iteration method, which was suggested in [8].

We note that the boundary condition for American options is the same as European options. Since the payoff is the same at expiry for both European and American options, the boundary condition at $t = T$ is the same. For the boundary condition at S = 0, as in the European case, we expect that the payoff will again be K, reduced in time at the risk free rate, so that $V(0,t) = Ke^{-r(T-t)}$. Finally, for the boundary as $S \to \infty$, we expect that the payoff be zero. This leads us to conclude that the boundary conditions for the American Put are (3). We use this method in the following example.

Example 2 Consider (2) and the condition (3) under the following parameters;

$$\begin{aligned} K = 50, \quad T = 3, \quad r = 0.05, \quad \sigma = 0.2, \\ M = 80, \quad N = 200, \quad S_{\max} = 150. \end{aligned} \tag{10}$$

The relaxation parameter in SOR method is $\omega = 1.2$. Implementing our method in a MATLAB code, the results are illustrated in Figure 2 for fractional American option with different values of α and β.

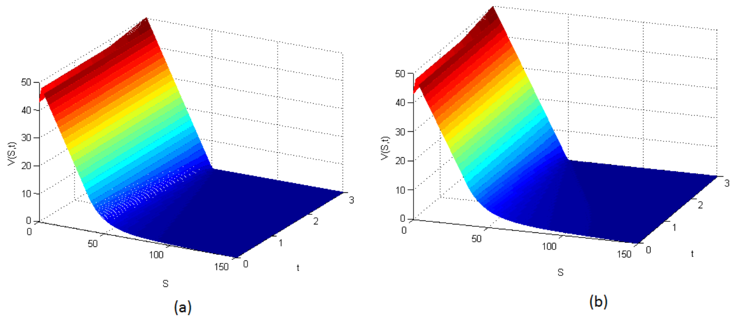

Fig. 2 American Put values calculated using the finite difference method for $\alpha = 0.9$ and (a) $\beta = 1.9$, (b) $\beta = 1.8$

5 Conclusion

We have implemented a numerical finite difference technique to solve the Black-Scholes European and, in particular, American options presented by FDE on time and asset. Knowing that fractional Black-Scholes American options do not have an exact solution, our proposed numerical methods have shown a good accuracy in presenting examples for different fractional order derivatives in time and asset.

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