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A High Order Finite Difference Method for Random Parabolic Partial Differential Equations

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Abstract In this paper, for the numerical approximation of random partial differential equations (RPDEs) of parabolic type, an explicit higher order finite difference scheme is constructed. In continuation the main properties of deterministic difference schemes, i.e. consistency, stability and convergency are developed for the random cases. It is shown that the proposed random difference scheme has these properties. Finally a numerical example is solved to illustrate the scheme of analysis.

Keywords Random partial differential equations · Consistency · Mean square stability · Convergence in mean square

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1 Introduction

Physical phenomena of interest in science and technology are very often theoretically simulated by means of models which correspond to partial differential equations (PDEs). These equations are in general nonlinear and, as such, their solution is usually a different task. Moreover, many times some of the parameters and initial data are not known with complete certainly due to lake of

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information, uncertainty in the measurements or incomplete knowledge of the mechanism themselves, and in practice any system undergoes perturbations from the surrounding ambient and, therefore, the behavior of the system itself is, in several circumstances, far away from the simple conditions of the ideal deterministic representation. To compensate this lake of information and to have a more realistic description of the system one introduces random inputs which may be a random variable or a stochastic process. This leads to RPDEs or a stochastic partial differential equations (SPDEs).
Some areas where SPDEs and RPDEs have been used extensively in modelling include chemistry, physics, engineering, mathematical biology and finance. In recent years, some of numerical methods for solving SPDEs like finite difference and finite element schemes, have been considered [1], [2], [3]. Also various numerical methods for RPDEs have been developed and analyzed [4], [5]. In this paper the random five points finite difference method is used to obtain an approximation solution for random parabolic partial differential equations. Mean square consistency of the random difference scheme for RPDE is established. The conditions for the mean square stability and mean square convergent of the proposed finite difference scheme are given. This paper is organized as follows: In next section we review some definitions relevant to mean square calculus. In section 3 we use an explicit finite difference scheme for random parabolic partial differential equations, and investigate consistency, stability and convergency of the resulting random difference scheme (RDS). Finally in the last section a numerical example is solved to illustrate the method of analysis.

2 A review of calculus in mean square

In order to develop main notions of deterministic difference schemes, that is consistency, stability and convergency to random case, firstly we give some definitions of calculus in mean square [7].

Definition 1 A sequence of real random variables $\{X_{nk} : n, k \geq 1\}$ whose second moments, i.e. $\mathbb{E}(X_{nk}^2)$ for all n, $k \geq 1$, are finite are called second–order random variables.

Let $\{X_{nk} : n,\ k \geq 1\}$, $\{Y_{nk} : n,\ k \geq 1\}$ be sequences of second order random variables over the same probability space. Using the notation

$$< X_{nk}, Y_{nk} >= \mathbb{E}(X_{nk}Y_{nk}),$$

and applying the Schwarz inequality it follows that:

$$\mathbb{E}((X_{nk} + Y_{nk})^2) < \infty, \qquad \mathbb{E}((cX_{nk})^2) = c^2\mathbb{E}(X_{nk}^2),$$

where c is real and finite. Therefore, the class of all second order random variables on a probability space constitute a linear vector space if all equivalent random variables are identified. One can easily show that $< X_{nk}, Y_{nk} >$

satisfies the inner product properties, hence, using the definition

$$\|X_{nk}\| =< X_{nk}, X_{nk} >^{\frac{1}{2}},$$

and using the properties of inner product, it can be shown that $\|X_{nk}\|$ possesses the norm properties. In addition, if we define the distance between X_{nk} and Y_{nk} by

$$d(X_{nk}, Y_{nk}) = \|X_{nk} - Y_{nk}\|, \tag{1}$$

then the distance $d(X_{nk}, Y_{nk})$ possesses the usual distance properties. We now give a definition of an L_2–space.

Definition 2 The linear vector space of second order random variables with inner product, the norm and the distance is defined in (1) is called an L_2–space.

Definition 3 (The convergence in mean square) A sequence of random variables $\{X_{n,k} : n,\, k \geq 1\}$ converges in mean square to a random variable X if

$$\lim_{n,k\to\infty} \|X_{n,k} - X\| = 0.$$

3 Finite difference approximation for random parabolic partial differential equations

Consider the following random parabolic partial differential equation

$$\begin{aligned} u_t(x,t) &= \beta u_{xx}(x,t), \qquad x \in [0, X], \quad t \in [0, T] \\ u(x,0) &= u_0(x), \qquad x \in [0, X] \\ u(0,t) &= u(X,t) = 0, \qquad t \in [0, T] \end{aligned} \tag{2}$$

where x is the space coordinate, t is the time variable and β is a positive random variable such that $\mathbb{E}(\beta^2) < \infty$. The random finite difference schemes are universal applicable numerical schemes for the solution of RPDEs. Basically these schemes discretize the continuous space and time into a evenly distributed grid system, and the values of the state variables are evaluated at each node of the grid. Introducing a uniform space grid Δx and a uniform time grid Δt, one gets a time–space lattice, for which one can attempt to approximate the solution of the above equation at the points of the lattice. Let u_k^n be a random variable defined at the point $(k\Delta x, n\Delta t)$ or the lattice point (k, n), where k and n are integers. The function u_k^n will be an approximation of the solution at the point $(k\Delta x, n\Delta t)$, where $u_k^0 = u_0(k\Delta x)$ and $u_0^n = u_X^n = 0$. The next step is to approximation problem on this grid. The method of undetermined coefficients enable us to find approximation to derivatives to any order desired. It can be shown that it is not possible to approximate u_{xx} to the fourth order using only the points $x = k\Delta x$ and $x = (k \pm 1)\Delta x$. As we develop the difference approximation of u_{xx}, we will see that it is possible to obtain a fourth order approximation of u_{xx} if we use the points $x = k\Delta x$, $x = (k\pm1)\Delta x$ and $x = (k \pm 2)\Delta x$ [8]. Thus we have an order Δx^4 approximation of u_{xx} as

$$u_{xx}(x,t) \approx \tfrac{1}{\Delta x^2}\left(-\tfrac{1}{12}u(x-2\Delta x,t) + \tfrac{4}{3}u(x-\Delta x,t) - \tfrac{5}{2}u(x,t)\right.$$
$$\left. + \tfrac{4}{3}u(x+\Delta x,t) - \tfrac{1}{12}u(x+2\Delta x,t)\right),$$

with the truncation error being $O(\Delta x^4)$. Again, we approximate the time derivative by the forward–difference expression

$$u_t(k\Delta x, n\Delta t) \approx \frac{u_k^{n+1} - u_k^n}{\Delta t}.$$

So (2) yields the approximation

$$u_k^{n+1} = u_k^n + \beta r\left(-\frac{1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n - \frac{5}{2}u_k^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n\right), \qquad (3)$$

where $r = \frac{\Delta t}{\Delta x^2}$, and the scheme is an $O(\Delta t) + O(\Delta x^4)$ approximation of (2) with $u_k^0 = u_0(k\Delta x)$ and $u_0^n = u_X^n = 0$.

Remark 1 For the proposed scheme, we assume that the random variable β is independent of the states u_k^n.

Essentially, it is extremely important for the solution of random difference scheme to converge to the solution of the RPDE. Consider a RPDE in the following form

$$Lv = G,$$

where L denotes the differential operator and $G \in L^2(\mathbb{R})$ is an inhomogeneity. Assuming u_k^n is the solution that is approximated by a random finite difference scheme denoted by L_k^n, and applying the random scheme to the RPDE, we have $L_k^n u_k^n = G_k^n$, where G_k^n is the approximation of the inhomogeneity. For consistency, stability and convergency, we will need a norm. Hence for a sequence $u = \{\ldots, u_{-1}, u_0, u_1, \ldots\}$, the sup–norm is defined as $\|u\|_\infty = \sqrt{\sup_k |u_k|^2}$ [6]. we refer to the paper by Roth [6], for the following definitions of random difference scheme.

Definition 4 (Consistency of an RDS) A random difference scheme $L_k^n u_k^n = G_k^n$ is pointwise consistent with the random partial differential equation $Lv = G$ at point (x,t), if for any continuously differentiable function $\Phi = \Phi(x,t)$, in mean square

$$\mathbb{E}\|\,(L\Phi - G)\,|_k^n - [L_k^n\Phi(k\Delta x, n\Delta t) - G_k^n]\,\|^2 \to 0,$$

as $\Delta x \to 0$, $\Delta t \to 0$, and $(k\Delta x, (n+1)\Delta t)$ converges to (x,t).

Consistency is a necessary criterion for a scheme to be convergent, but it is not sufficient. The second important property is stability.

Definition 5 (Stability of an RDS) A random difference scheme is said to be stable with respect to a norm in mean square if there exist some positive constants $\overline{\Delta x_0}$ and $\overline{\Delta t_0}$ and non–negative constants K and γ such that

$$\mathbb{E}\|u^{n+1}\|^2 \leq K e^{\gamma t}\mathbb{E}\|u^0\|^2,$$

for all $0 \leq t = (n+1)\Delta t$, $0 \leq \Delta x \leq \overline{\Delta x_0}$, and $0 \leq \Delta t \leq \overline{\Delta t_0}$, where

$$u^{n+1} = \left(\ldots, u_{k-2}^{n+1}, u_{k-1}^{n+1}, u_k^{n+1}, u_{k+1}^{n+1}, u_{k+2}^{n+1}, \ldots\right)^T.$$

Definition 6 (Convergence of an RDS) A random difference scheme $L_k^n u_k^n = G_k^n$ approximating the random partial differential equation $Lv = G$ is convergent in mean square at time $t = (n+1)\Delta t$, if $\mathbb{E}|u_k^n - v|^2 \to 0$, as $\Delta x \to 0$, $\Delta t \to 0$, and $(k\Delta x, n\Delta t) \to (x, t)$.

Theorem 1 *The random difference scheme (3) is consistent in mean square in the sense of Definition 4.*

Proof Let $\Phi(x,t)$ be a smooth function. Then we have:

$$L(\Phi)|_k^n = \Phi(k\Delta x,(n+1)\Delta t) - \Phi(k\Delta x, n\Delta t) - \beta\int_{n\Delta t}^{(n+1)\Delta t}\Phi_{xx}(k\Delta x, s)\mathrm{d}s,$$

and

$$\begin{aligned}L_k^n\Phi = {}& \Phi(k\Delta x,(n+1)\Delta t) - \Phi(k\Delta x, n\Delta t) - \beta\frac{\Delta t}{\Delta x^2}\Big(-\frac{1}{12}\Phi((k-2)\Delta x, n\Delta t)\\ &+\frac{4}{3}\Phi((k-1)\Delta x, n\Delta t) - \frac{5}{2}\Phi(k\Delta x, n\Delta t) + \frac{4}{3}\Phi((k+1)\Delta x, n\Delta t)\\ &-\frac{1}{12}\Phi((k+2)\Delta x, n\Delta t)\Big).\end{aligned}$$

Therefore, in mean square, we obtain:

$$\begin{aligned}\mathbb{E}|L(\Phi)|_k^n - L_k^n\Phi|^2 = {}& \mathbb{E}(\beta^2)\Bigg|\int_{n\Delta t}^{(n+1)\Delta t}\Big(\Phi_{xx}(k\Delta x, s) - \frac{1}{\Delta x^2}\Big[-\frac{1}{12}\Phi((k-2)\Delta x, n\Delta t)\\ &+\frac{4}{3}\Phi((k-1)\Delta x, n\Delta t) - \frac{5}{2}\Phi(k\Delta x, n\Delta t) + \frac{4}{3}\Phi((k+1)\Delta x, n\Delta t)\\ &-\frac{1}{12}\Phi((k-2)\Delta x, n\Delta t)\Big]\Big)\,\mathrm{d}s\Bigg|^2.\end{aligned}$$

Since $\Phi(x,t)$ is a deterministic function, we have:

$$\mathbb{E}|L(\Phi)|_k^n - L_k^n\Phi|^2 \to 0,$$

as n, $k \to \infty$. This proves the consistency.

Theorem 2 *The random difference scheme (3) with* $t = (n+1)\Delta t$ *and* $0 \leq \beta\frac{\Delta t}{\Delta x^2} := \beta r \leq \frac{2}{5}$, *is stable with respect to* $\|\cdot\|_\infty = \sqrt{\sup_k|\cdot|^2}$.

Proof Applying $\mathbb{E}|\cdot|^2$ in (3), we will have:

$$\begin{aligned}
\mathbb{E}|u_k^{n+1}|^2 &= \mathbb{E}\left|\left(1-\frac{5}{2}r\beta\right)u_k^n + r\beta\left(-\frac{1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n\right)\right|^2 \\
&\leq \mathbb{E}\left(1-\frac{5}{2}r\beta\right)^2\mathbb{E}|u_k^n|^2 + \frac{1}{144}\mathbb{E}(r\beta)^2\mathbb{E}|u_{k-2}^n|^2 + \frac{16}{9}\mathbb{E}(r\beta)^2\mathbb{E}|u_{k-1}^n|^2 \\
&\quad + \frac{16}{9}\mathbb{E}(r\beta)^2\mathbb{E}|u_{k+1}^n|^2 + \frac{1}{144}\mathbb{E}(r\beta)^2\mathbb{E}|u_{k+2}^n|^2 \\
&\quad + 2\mathbb{E}\left(\left(1-\frac{5}{2}r\beta\right)\frac{r\beta}{12}\right)\mathbb{E}|u_k^n u_{k-2}^n| + 2\mathbb{E}\left(\left(1-\frac{5}{2}r\beta\right)\frac{4}{3}r\beta\right)\mathbb{E}|u_k^n u_{k-1}^n| \\
&\quad + 2\mathbb{E}\left(\left(1-\frac{5}{2}r\beta\right)\frac{4}{3}r\beta\right)\mathbb{E}|u_k^n u_{k+1}^n| + 2\mathbb{E}\left(\left(1-\frac{5}{2}r\beta\right)\frac{r\beta}{12}\right)\mathbb{E}|u_k^n u_{k+2}^n| \\
&\quad + 2\cdot\left(\frac{1}{12}\cdot\frac{4}{3}\right)\mathbb{E}(r\beta)^2\mathbb{E}|u_{k-2}^n u_{k-1}^n| + 2\cdot\left(\frac{1}{12}\cdot\frac{4}{3}\right)\mathbb{E}(r\beta)^2\mathbb{E}|u_{k-2}^n u_{k+1}^n| \\
&\quad + \frac{2}{144}\mathbb{E}(r\beta)^2\mathbb{E}|u_{k-2}^n u_{k+2}^n| + \frac{32}{9}\mathbb{E}(r\beta)^2 E|u_{k-1}^n u_{k+1}^n| \\
&\quad + 2\cdot\left(\frac{1}{12}\cdot\frac{4}{3}\right)\mathbb{E}(r\beta)^2\mathbb{E}|u_{k-1}^n u_{k+2}^n| + 2\cdot\left(\frac{1}{12}\cdot\frac{4}{3}\right)\mathbb{E}(r\beta)^2\mathbb{E}|u_{k+1}^n u_{k+2}^n| \\
&\leq \left(1+\frac{1}{9}r^2\mathbb{E}(\beta^2) + \frac{2}{3}r\mathbb{E}(\beta)\right)\sup_k \mathbb{E}|u_k^n|^2.
\end{aligned}$$

It is enough to select λ such that $\frac{1}{9}r^2\mathbb{E}(\beta^2) + \frac{2}{3}r\mathbb{E}(\beta) \leq \lambda^2 \Delta t$ holds, for all k. Therefore

$$\sup_k \mathbb{E}|u_k^{n+1}|^2 \leq (1+\lambda^2\Delta t)\sup_k \mathbb{E}|u_k^n|^2 \leq \cdots \leq (1+\lambda^2\Delta t)^{n+1}\sup_k \mathbb{E}|u_k^0|^2,$$

and by substituting Δt with $\frac{t}{n+1}$,

$$\begin{aligned}
\mathbb{E}\|u^{n+1}\|_\infty^2 &\leq \left(1+\frac{\lambda^2 t}{n+1}\right)^{n+1}\mathbb{E}\|u^0\|_\infty^2 \\
&\leq e^{\lambda^2 t}\mathbb{E}\|u^0\|_\infty^2.
\end{aligned}$$

Hence, the random difference scheme is conditionally stable with $K = 1$ and $\gamma = \lambda^2$.

Theorem 3 *The random scheme (3) is convergent for the $\|\cdot\|_\infty$–norm, for $0 \leq r\beta \leq \frac{2}{5}$.*

Proof The random finite difference scheme is given by

$$u_k^{n+1} = u_k^n + \beta r\left(-\frac{1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n - \frac{5}{2}u_k^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n\right). \tag{4}$$

The solution v_k^{n+1} can be represented by the Taylor expansion $v_{xx}(x,s)$ with respect to the space variable as

$$\begin{aligned} v_k^{n+1} &= v_k^n + \beta \int_{n\Delta t}^{(n+1)\Delta t} v_{xx}(x_k, s)\mathrm{d}s \\ &= v_k^n + \beta \int_{n\Delta t}^{(n+1)\Delta t} \left(\frac{1}{\Delta x^2} \left(-\frac{1}{12} v_{k-2}^n + \frac{4}{3} v_{k-1}^n - \frac{5}{2} v_k^n + \frac{4}{3} v_{k+1}^n - \frac{1}{12} v_{k+2}^n \right) \right. \\ &\left. -\frac{1}{90} v^{(6)}((k+\nu)\Delta x, s)\Delta x^4 \right) \mathrm{d}s, \end{aligned}$$

where $\nu \in (0,1)$. With $z_k^n = v_k^n - u_k^n$ we get

$$\begin{aligned} z_k^{n+1} &= v_k^{n+1} - u_k^{n+1} \\ &= z_k^n + \beta \int_{n\Delta t}^{(n+1)\Delta t} \left(\frac{1}{\Delta x^2} \left(-\frac{1}{12} z_{k-2}^n + \frac{4}{3} z_{k-1}^n - \frac{5}{2} z_k^n + \frac{4}{3} z_{k+1}^n - \frac{1}{12} z_{k+2}^n \right) \right. \\ &\left. -\frac{1}{90} v^{(6)}((k+\nu)\Delta x, s)\Delta x^4 \right) \mathrm{d}s. \end{aligned}$$

Then, we have:

$$\begin{aligned} \mathbb{E}|z_k^{n+1}|^2 &= \mathbb{E} \left| z_k^n + \beta \int_{n\Delta t}^{(n+1)\Delta t} \left(\frac{1}{\Delta x^2} \left(-\frac{1}{12} z_{k-2}^n + \frac{4}{3} z_{k-1}^n - \frac{5}{2} z_k^n \right.\right.\right. \\ &\left.\left.\left. +\frac{4}{3} z_{k+1}^n - \frac{1}{12} z_{k+2}^n \right) - \frac{1}{90} v^{(6)}((k+\nu)\Delta x, s)\Delta x^4 \right) \mathrm{d}s \right|^2 \\ &= \mathbb{E} \left| \left(1 - \frac{5}{2} r\beta\right) z_k^n + r\beta \left(-\frac{1}{12} z_{k-2}^n + \frac{4}{3} z_{k-1}^n + \frac{4}{3} z_{k+1}^n - \frac{1}{12} z_{k+2}^n \right) \right. \\ &\left. -\frac{1}{90} v^{(6)}((k+\nu)\Delta x, s)\Delta t \Delta x^4 \right|^2. \end{aligned}$$

Assuming $0 \le r\beta \le \frac{2}{5}$, introducing the notation $A_k = \frac{1}{90} v^{(6)}((k+\nu)\Delta x, s) < \infty$ and also using the inequality

$$\mathbb{E}|X+Y|^2 \le 2\mathbb{E}|X|^2 + 2\mathbb{E}|Y|^2,$$

we have:

$$\begin{aligned} \mathbb{E}|z_k^{n+1}|^2 &\le 2\mathbb{E} \left| \left(1 - \frac{5}{2} r\beta\right) z_k^n + r\beta \left(-\frac{1}{12} z_{k-2}^n + \frac{4}{3} z_{k-1}^n + \frac{4}{3} z_{k+1}^n - \frac{1}{12} z_{k+2}^n \right) \right|^2 \\ &+2(A_k \Delta x^4 \Delta t)^2 \\ &\le 2 \left(1 + \frac{1}{9} r^2 \mathbb{E}(\beta^2) + \frac{2}{3} r\mathbb{E}(\beta) \right) \sup_k \mathbb{E}|z_k^n|^2 + 2(A_k \Delta x^4 \Delta t)^2 \\ &= \left(1 + 1 + \frac{2}{9} r^2 \mathbb{E}(\beta^2) + \frac{4}{3} r\mathbb{E}(\beta) \right) \sup_k \mathbb{E}|z_k^n|^2 + 2(A_k \Delta x^4 \Delta t)^2. \end{aligned}$$

With selecting σ such that $1 + \frac{2}{9}r^2\mathbb{E}(\beta^2) + \frac{4}{3}r\mathbb{E}(\beta) \leq \sigma^2\Delta t$, we have:

$$\mathbb{E}|z_k^{n+1}|^2 \leq (1 + \sigma^2\Delta t)\sup_k \mathbb{E}|z_k^n|^2 + 2(A_k\Delta x^4\Delta t)^2,$$

for all k. This yields

$$\begin{aligned}\mathbb{E}\|z^{n+1}\|_\infty^2 &\leq (1 + \sigma^2\Delta t)\mathbb{E}\|z^n\|_\infty^2 + M_k(\Delta x^4\Delta t)^2 \\ &\leq (1 + \sigma^2\Delta t)^2\mathbb{E}\|z^{n-1}\|_\infty^2 + (1 + \sigma^2\Delta t)M_k(\Delta x^4\Delta t)^2 + M_k(\Delta x^4\Delta t)^2 \\ &\vdots \\ &\leq (1 + \sigma^2\Delta t)^{n+1}E\|z^0\|_\infty^2 \\ &\quad + \Big((1 + \sigma^2\Delta t)^n + \cdots + (1 + \sigma^2\Delta t) + 1\Big)M_k(\Delta x^4\Delta t)^2,\end{aligned}$$

where $M_k = 2A_k^2$. Since $\Delta t = \frac{t}{n+1}$, we get

$$\begin{aligned}\mathbb{E}\|z^{n+1}\|_\infty^2 &\leq e^{\sigma^2 t}\mathbb{E}\|z^0\|_\infty^2 + \sum_{i=0}^{n}(1 + \sigma^2\Delta t)^i M_k(\Delta x^4\Delta t)^2 \\ &= M_k(\Delta x^4\Delta t)^2\sum_{i=0}^{n}(1 + \sigma^2\Delta t)^i \\ &\leq M_k(\Delta x^4\Delta t)^2\sum_{i=0}^{n}(1 + \sigma^2\Delta t)^{n+1} = M_k(\Delta x^4\Delta t)^2(n+1)(1 + \sigma^2\Delta t)^{n+1} \\ &\leq M_k(\Delta x^4)^2\Delta t\cdot(n+1)\Delta t e^{\sigma^2 t} = M_k(\Delta x^4)^2\Delta t\cdot t e^{\sigma^2 t} \to 0,\end{aligned}$$

as $\Delta x \to 0$ and $\Delta t \to 0$. Hence the random difference scheme is convergent in mean square sense.

4 Numerical illustration

In this section, the performance of the explicit finite difference scheme described in the previous section on one problem, for which the exact solution is known, is considered.

Example 1 Consider the random parabolic partial differential equation

$$u_t(x,t) = \beta u_{xx}(x,t), \qquad x \in [0,X], \quad t \in [0,T]$$

with initial condition

$$u(x,0) = \sin\pi x, \qquad x \in [0,X]$$

and the homogeneous boundary conditions

$$u(0,t) = u(X,t) = 0, \qquad t \in [0,T].$$

The exact solution given by

$$u(x,t) = e^{-\beta\pi^2 t}\sin \pi x.$$

Consider, a uniform mesh with stepsize Δx and Δt on x–axis and t–axis, where $\Delta x = \frac{X}{M}$, $\Delta t = \frac{T}{N}$, and $M,\ N > 0$. On this mesh, the difference scheme for this problem is

$$\begin{aligned} u_k^{n+1} &= \left(1 - \frac{5}{2}\rho\right) u_k^n + \rho\left(-\frac{1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n\right), \quad (5)\\ u_k^0 &= \sin \pi x_k = \sin(k\pi\Delta x),\\ u_0^n &= u_X^n = 0, \end{aligned}$$

where $\rho = \beta\frac{\Delta t}{\Delta x^2}$, $x_k = k\Delta x$ and $t_n = n\Delta t$. Here we assume that $u_{-1}^n = u_{M+1}^n = 0$. From (5) we have:

$$\begin{aligned} u_0^0 &= 0,\\ u_1^0 &= \sin(\Delta x\pi),\\ u_2^0 &= \sin(2\Delta x\pi). \end{aligned}$$

Then

$$u_k^0 = \sin(k\Delta x\pi).$$

From (5) it follows that

$$\begin{aligned} u_1^1 &= \left(1 - \frac{5}{2}\rho\right) u_1^0 + \rho\left(-\frac{1}{12}u_{-1}^0 + \frac{4}{3}u_0^0 + \frac{4}{3}u_2^0 - \frac{1}{12}u_3^0\right)\\ &= \left(1 - \frac{5}{2}\rho\right)\sin(\Delta x\pi) + \rho\left(\frac{4}{3}\sin(2\Delta x\pi) - \frac{1}{12}\sin(3\Delta x\pi)\right),\\ u_2^1 &= \left(1 - \frac{5}{2}\rho\right) u_2^0 + \rho\left(-\frac{1}{12}u_0^0 + \frac{4}{3}u_1^0 + \frac{4}{3}u_3^0 - \frac{1}{12}u_4^0\right)\\ &= \left(1 - \frac{5}{2}\rho\right)\sin(2\Delta x\pi) + \rho\left(\frac{4}{3}\sin(\Delta x\pi) + \frac{4}{3}\sin(3\Delta x\pi) - \frac{1}{12}\sin(4\Delta x\pi)\right). \end{aligned}$$

Finally, we obtain

$$
\begin{aligned}
u_k^1 &= \left(1-\frac{5}{2}\rho\right)u_k^0+\rho\left(-\frac{1}{12}u_{k-2}^0+\frac{4}{3}u_{k-1}^0+\frac{4}{3}u_{k+1}^0-\frac{1}{12}u_{k+2}^0\right)\\
&= \left(1-\frac{5}{2}\rho\right)\sin(k\Delta x\pi)+\rho\Big(-\frac{1}{12}\sin((k-2)\Delta x\pi)+\frac{4}{3}\sin((k-1)\Delta x\pi)\\
&\quad+\frac{4}{3}\sin((k+1)\Delta x\pi)-\frac{1}{12}\sin((k+2)\Delta x\pi)\Big)\\
&= \left(1-\frac{5}{2}\rho\right)\sin(k\Delta x\pi)+\rho\Big(-\frac{1}{12}(2\sin(k\Delta x\pi)\cos(2\Delta x\pi))\\
&\quad+\frac{4}{3}(2\sin(k\Delta x\pi)\cos(\Delta x\pi))\Big)\\
&= \sin(k\Delta x\pi)\left(\left(1-\frac{5}{2}\rho\right)-\frac{\rho}{6}\cos(2\Delta x\pi)+\frac{8\rho}{3}\cos(\Delta x\pi)\right)\\
&= \sin(k\Delta x\pi)\left(1+\frac{\rho}{6}-\frac{8\rho}{3}-\frac{\rho}{6}\cos(2\Delta x\pi)+\frac{8\rho}{3}\cos(\Delta x\pi)\right)\\
&= \sin(k\Delta x\pi)\left(1+\frac{\rho}{6}(1-\cos(2\Delta x\pi))-\frac{8\rho}{3}(1-\cos(\Delta x\pi))\right)\\
&= \sin(k\Delta x\pi)\left(1+\frac{\rho}{6}(2\sin^2(\Delta x\pi))-\frac{8\rho}{3}\left(2\sin^2\left(\frac{\Delta x\pi}{2}\right)\right)\right)\\
&= \sin(k\Delta x\pi)\left(1+\frac{\rho}{3}\left(\frac{e^{i\Delta x\pi}-e^{-i\Delta x\pi}}{2i}\right)^2-\frac{16\rho}{3}\left(\frac{e^{\frac{i\Delta x\pi}{2}}-e^{\frac{-i\Delta x\pi}{2}}}{2i}\right)^2\right)\\
&= \sin(k\Delta x\pi)\left(1-\frac{\rho}{12}\left(e^{2i\Delta x\pi}-2+e^{-2i\Delta x\pi}\right)+\frac{16\rho}{12}\left(e^{i\Delta x\pi}-2+e^{-i\Delta x\pi}\right)\right)\\
&= \sin(k\Delta x\pi)\Big[1-\frac{\rho}{12}\left(-\frac{2(2\Delta x\pi)^2}{2!}+\frac{2(2\Delta x\pi)^4}{4!}-\frac{2(2\Delta x\pi)^6}{6!}+\cdots\right)\\
&\quad+\frac{16\rho}{12}\left(-\frac{2(\Delta x\pi)^2}{2!}+\frac{2(\Delta x\pi)^4}{4!}-\frac{2(\Delta x\pi)^6}{6!}+\cdots\right)\Big]\\
&= \sin(k\Delta x\pi)\left(1-\frac{\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(2\Delta x\pi)^{2j}}{(2j)!}+\frac{16\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(\Delta x\pi)^{2j}}{(2j)!}\right).
\end{aligned}
$$

Therefore, we have:

$$
u_k^1=\sin(k\Delta x\pi)\left(1-\frac{\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(2\Delta x\pi)^{2j}}{(2j)!}+\frac{16\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(\Delta x\pi)^{2j}}{(2j)!}\right).
$$

Also

$$\begin{aligned}
u_1^2 &= \left(1-\frac{5}{2}\rho\right)u_1^1+\rho\left(-\frac{1}{12}u_{-1}^1+\frac{4}{3}u_0^1+\frac{4}{3}u_2^1-\frac{1}{12}u_3^1\right)\\
&= \left(1-\frac{5}{2}\rho\right)\sin(\Delta x\pi)\left(1-\frac{\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(2\Delta x\pi)^{2j}}{(2j)!}+\frac{16\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(\Delta x\pi)^{2j}}{(2j)!}\right)\\
&\quad+\rho\Big(\frac{4}{3}\sin(2\Delta x\pi)\Big(1-\frac{\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(2\Delta x\pi)^{2j}}{(2j)!}+\frac{16\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(\Delta x\pi)^{2j}}{(2j)!}\Big)\\
&\quad-\frac{1}{12}\sin(3\Delta x\pi)\Big(1-\frac{\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(2\Delta x\pi)^{2j}}{(2j)!}+\frac{16\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(\Delta x\pi)^{2j}}{(2j)!}\Big)\Big),\\
u_k^2 &= \left(1-\frac{5}{2}\rho\right)u_k^1+\rho\left(-\frac{1}{12}u_{k-2}^1+\frac{4}{3}u_{k-1}^1+\frac{4}{3}u_{k+1}^1-\frac{1}{12}u_{k+2}^1\right)\\
&= \left(1-\frac{5}{2}\rho\right)\sin(k\Delta x\pi)\Big(1-\frac{\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(2\Delta x\pi)^{2j}}{(2j)!}+\frac{16\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(\Delta x\pi)^{2j}}{(2j)!}\Big)\\
&\quad+\rho\Big(1-\frac{\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(2\Delta x\pi)^{2j}}{(2j)!}+\frac{16\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(\Delta x\pi)^{2j}}{(2j)!}\Big)\\
&\quad\Big(-\frac{1}{12}(\sin((k-2)\Delta x\pi)+\sin((k+2)\Delta x\pi))\\
&\quad+\frac{4}{3}(\sin((k-1)\Delta x\pi)+\sin((k+1)\Delta x\pi))\Big)\\
&= \left(1-\frac{5}{2}\rho\right)\sin(k\Delta x\pi)\Big(1-\frac{\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(2\Delta x\pi)^{2j}}{(2j)!}+\frac{16\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(\Delta x\pi)^{2j}}{(2j)!}\Big)\\
&\quad+\rho\Big(1-\frac{\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(2\Delta x\pi)^{2j}}{(2j)!}+\frac{16\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(\Delta x\pi)^{2j}}{(2j)!}\Big)\\
&\quad\Big(-\frac{1}{6}\sin(k\Delta x\pi)\cos(2\Delta x\pi)+\frac{8}{3}\sin(k\Delta x\pi)\cos(\Delta x\pi)\Big)\\
&= \sin(k\Delta x\pi)\left(1-\frac{\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(2\Delta x\pi)^{2j}}{(2j)!}+\frac{16\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(\Delta x\pi)^{2j}}{(2j)!}\right)^2 \qquad (6)\\
\left(\left(1-\frac{5}{2}\rho\right)\right) &= \sin(k\Delta x\pi)\left(1-\frac{\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(2\Delta x\pi)^{2j}}{(2j)!}+\frac{16\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(\Delta x\pi)^{2j}}{(2j)!}\right)^2.
\end{aligned}$$

Finally, the numerical solution for this problem is

$$u_k^n=\sin(k\Delta x\pi)\left(1-\frac{\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(2\Delta x\pi)^{2j}}{(2j)!}+\frac{16\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(\Delta x\pi)^{2j}}{(2j)!}\right)^n.$$

Hence

$$
\begin{aligned}
u_k^n - u = {} & \sin(k\Delta x\pi)\left(1-\frac{\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(2\Delta x\pi)^{2j}}{(2j)!}+\frac{16\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(\Delta x\pi)^{2j}}{(2j)!}\right)^n \\
& -e^{-\beta\pi^2 t}\sin(\pi x),
\end{aligned}
$$

$$
\begin{aligned}
|u_k^n - u|^2 = {} & \Big|\sin(k\Delta x\pi)\left(1-\frac{\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(2\Delta x\pi)^{2j}}{(2j)!}+\frac{16\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(\Delta x\pi)^{2j}}{(2j)!}\right)^n \\
& -e^{-\beta\pi^2 t}\sin(\pi x)\Big|^2 \\
= {} & \left(\sin(k\Delta x\pi)\left(1-\frac{\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(2\Delta x\pi)^{2j}}{(2j)!}+\frac{16\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(\Delta x\pi)^{2j}}{(2j)!}\right)^n\right)^2 \\
& -2\sin(k\Delta x\pi)\left(1-\frac{\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(2\Delta x\pi)^{2j}}{(2j)!}+\frac{16\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(\Delta x\pi)^{2j}}{(2j)!}\right)^n \\
& (e^{-\beta\pi^2 t}\sin(\pi x)) + (e^{-\beta\pi^2 t}\sin(\pi x))^2 \\
= {} & \sin^2(k\Delta x\pi)\left(1-\frac{\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(2\Delta x\pi)^{2j}}{(2j)!}+\frac{16\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(\Delta x\pi)^{2j}}{(2j)!}\right)^{2n} \\
& -2\sin(k\Delta x\pi)\left(1-\frac{\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(2\Delta x\pi)^{2j}}{(2j)!}+\frac{16\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(\Delta x\pi)^{2j}}{(2j)!}\right)^n \\
& (e^{-\beta\pi^2 t}\sin(\pi x)) + e^{-2\beta\pi^2 t}\sin^2(\pi x).
\end{aligned}
$$

So, we get

$$
\begin{aligned}
\mathbb{E}|u_k^n - u|^2 = {} & \mathbb{E}\left(\sin^2(k\Delta x\pi)\left(1-\frac{\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(2\Delta x\pi)^{2j}}{(2j)!}+\frac{16\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(\Delta x\pi)^{2j}}{(2j)!}\right)^{2n}\right) \\
& -2\mathbb{E}\Big(\sin(k\Delta x\pi)\left(1-\frac{\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(2\Delta x\pi)^{2j}}{(2j)!}+\frac{16\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(\Delta x\pi)^{2j}}{(2j)!}\right)^n \\
& (e^{-\beta\pi^2 t}\sin(\pi x))\Big) + \mathbb{E}\left(e^{-2\beta\pi^2 t}\sin^2(\pi x)\right).
\end{aligned}
$$

Hence at time $t = (n+1)\Delta t$ we obtain:

$$\begin{aligned}\mathbb{E}|u_k^n - u|^2 = \mathbb{E}\left(\sin^2(k\Delta x\pi)\left(1 - \frac{\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(2\Delta x\pi)^{2j}}{(2j)!} + \frac{16\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(\Delta x\pi)^{2j}}{(2j)!}\right)^{2n}\right) \\ -2\mathbb{E}\Big(\sin(k\Delta x\pi)\left(1 - \frac{\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(2\Delta x\pi)^{2j}}{(2j)!} + \frac{16\rho}{6}\sum_{j=1}^{\infty}(-1)^j\frac{(\Delta x\pi)^{2j}}{(2j)!}\right)^{n} \\ (e^{-\beta\pi^2(n+1)\Delta t}\sin(\pi x))\Big) + \mathbb{E}\left(e^{-2\beta\pi^2(n+1)\Delta t}\sin^2(\pi x)\right).\end{aligned}$$

By taking the limit as $\Delta x \to 0$, $\Delta t \to 0$, $(k\Delta x, (n+1)\Delta t) \to (x,t)$ and under condition $0 \le \rho \le \frac{2}{5}$, then we obtain $\mathbb{E}|u_k^n - u|^2 \to 0$.

5 Conclusion

In this paper, we solve random parabolic partial differential equation with a high order random finite difference scheme. Mean square consistency of the random finite difference scheme is established. Sufficient conditions for the mean square stability of the proposed random difference scheme are given. In continuation the convergence of the solution scheme to the exact one is proved. Finally an example is solved to illustrate the method of analysis.

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