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On the Exact Solution for Nonlinear Partial Differential Equations

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Abstract In this study, we aim to construct a traveling wave solution for nonlinear partial differential equations. In this regards, a cosine-function method is used to find and generate the exact solutions for three different types of nonlinear partial differential equations such as general regularized long wave equation (GRLW), general Korteweg-de Vries equation (GKDV) and general equal width wave equation (GEWE) which are the major soliton equations.

Keywords Nonlinear PDEs · GRLW · GKDV · GEWE · Cosine-function

Mathematics Subject Classification (2010) 35K05 · 65M32

1 Introduction

From both theoretical and practical points of view, the investigation and study of numerical approaches for the solution of different differential equations has been an intense period of activity over the last 50 years. Many modifications in numerical techniques and algorithms, together with the rapid advances in computer technology, have meant that many of the partial differential equations arising from engineering and scientific applications, which were previously intractable, can now, be easily solved [1]. Foe example, the well-known finite difference methods can approximate the differential operators and accordingly it can solve the difference equations. In fact, in finite element method the continuous domain is represented as a collection of a finite number N of

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subdomains known as elements. The collection of elements is called the finite element mesh. For time dependent problems, the differential equations are approximated by the finite element method to obtain a set of ordinary differential equations in time. These differential equations are solved approximately by finite difference methods or other methods. In all finite difference and finite elements it is necessary to have a boundary and initial conditions. But the Adomian decomposition method, which has been developed by George Adomian [2] depends only on the initial conditions to obtain solution in series form which almost converges to the exact solutions of the problem.

The main goal of this paper is to apply the cosine-function method to obtain the exact solutions for the three different types of nonlinear partial differential equations like general regularized long wave equation GRLW, general Korteweg-de Vries equation GKDV, general equal width wave equation GEWE, which are all the significant soliton equations.

2 A brief survey on the method

Consider the nonlinear partial differential equation in the form

$$F(u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{xy}, u_{yy}, \dots) = 0, \tag{1}$$

where $u(x, y, t)$ is a traveling wave solution of nonlinear partial differential equation (1). We use the transformations,

$$u(x, y, t) = f(\xi), \tag{2}$$

where $\xi = x + y - ct$. This enables us to use the following changes

$$\frac{\partial}{\partial t}(.) = -c\frac{d}{d\xi}(.), \qquad \frac{\partial}{\partial x}(.) = \frac{d}{d\xi}(.), \qquad \frac{\partial}{\partial y}(.) = \frac{d}{d\xi}(.), \quad \dots \tag{3}$$

using Eq. (3) to transfer the nonlinear partial differential equation Eq. (1) to nonlinear ordinary differential equation

$$Q(f, f', f'', f''', \dots) = 0. \tag{4}$$

The ordinary differential equation (4) is then integrated as long as all terms contain derivatives, where we neglect the integration constants.
The solutions of many nonlinear equations can be expressed in the form

$$f(\xi) = \begin{cases} \lambda \cos^{\beta}(\mu\xi) & |\xi| \leqslant \dfrac{\pi}{2\mu}, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Where λ, μ and β are parameters to be determined, μ and c are wave number and wave speed, respectively. we use

$$f(\xi) = \lambda \cos^{\beta}(\mu\xi),$$

$$f'(\xi) = \frac{df(\xi)}{d\xi} = \tag{6}$$

$$-\lambda\beta\mu\cos^{\beta-1}(\mu\xi)\sin(\mu\xi),$$

$$f''(\xi) = \frac{d^2f(\xi)}{d\xi^2} =$$

$$-\lambda\beta\mu^2\cos^{\beta}(\mu\xi) + \lambda\mu^2\beta(\beta-1)cos^{\beta-2}(\mu\xi) - \lambda\mu^2\beta\ (\beta-1)\cos^{\beta}(\mu\xi).$$

$$\vdots$$

Substituting Eq. (6) in to the nonlinear ordinary differential equation Eq. (4) gives a trigonometric equation of $cos^{\alpha}(\mu\xi)$ terms. The exponents of each pair of cosine to determine α.

Then we collect all terms with the same power in $cos^{\beta}(\mu\xi)$ and put to zero their coefficients to get a system of algebraic equations among the unknowns β, λ and μ. Now the problem is reduced to a system of algebraic equations that can be solved to obtain the unknown parameters β, λ and μ. Hence, the solution considered in Eq. (5) is obtained. The above analysis yields the following theorem

Theorem 1 *The exact analytical solution of the nonlinear partial differential equation* (1) *can be determined in the form Eq.* (5) *where all constants found from the algebraic equations after it's solutions.*

3 Experimental Results and Applications

In this section in order to illustrate the effectiveness of the method three different examples in mathematical are chosen as follows

Example 1 The general regularized long wave equation (GRLW)
Let us first consider the following problem: find functions $u(x,t)$ satisfying the equation in the form

$$u_t + u_x + \varepsilon\ u^p\ u_x - vu_{xxt} = 0. \tag{7}$$

By using the wave variable $\xi = x - ct$ and $u(x,t) = f(\xi)$ than the equation (7) becomes

$$-cf' + f' + \varepsilon f^p f' + cvf''' = 0. \tag{8}$$

Since all the terms in (8) contain derivatives, integrating it once, we get

$$(1-c)f + \frac{\varepsilon\ f^{p+1}}{p+1} + cvf'' = 0. \tag{9}$$

We substitute equation (6) to (9) to obtain

$$(1-c)\lambda\cos^{\beta}(\mu\xi) + \frac{\varepsilon}{p+1}(\lambda^{p+1}\cos^{\beta(p+1)}(\mu\xi)) - cv\lambda\beta\mu^2\cos^{\beta}(\mu\xi) \tag{10}$$

$$+cv\lambda\mu^2\beta\ (\beta-1)\cos^{\beta-2}\ (\mu\xi) - cv\lambda\mu^2\beta\ (\beta-1)\cos^{\beta}\ (\mu\xi) = 0.$$

By equating the exponents and the coefficients of each pair of the cosine functions, we get the following system of algebraic equations

$$\begin{gathered}(p+1)\beta = \beta - 2,\\ (1-c)\lambda - cv\lambda\beta\mu^2 - cv\lambda\mu^2\beta\ (\beta-1) = 0,\\ \frac{\varepsilon\lambda^{p+1}}{p+1} + cv\lambda\mu^2\beta\ (\beta-1) = 0.\end{gathered} \tag{11}$$

Using Mathematica, we solve this system of equations and thereby obtain

$$\begin{gathered}\beta = \frac{-2}{p},\\ \mu = \pm\frac{p\sqrt{1-c}}{2\sqrt{vc}},\\ \lambda = 2^{-\frac{1}{p}}\left(\frac{-2+2c-3p+3cp-p^2+cp^2}{\varepsilon}\right)^{\frac{1}{p}}.\end{gathered} \tag{12}$$

Finally substituting equation (12) in to equation (5), we get

$$u(x,t) = 2^{\frac{-1}{p}}\left(\frac{-2+2c-3p+3cp-p^2+cp^2}{\varepsilon}\right)^{\frac{1}{p}}\cos^{\frac{-2}{p}}\left(\pm\frac{p\sqrt{1-c}}{2\sqrt{vc}}(x-ct)\right), \tag{13}$$

which is the exact soliton solution of the GRLW equation.

Example 2 The general Korteweg-de Vries (GKDV) Equation
In this example, we consider the traveling solution for equation takes the form

$$u_x + \varepsilon\ u^p u_x + \gamma\ u_{xx} = 0. \tag{14}$$

We follow the same procedures as applied in to the previous example (1) and obtain the following system of equations

$$\begin{gathered}(p+1)\beta = \beta - 2,\\ -c\lambda - \gamma\lambda\beta\mu^2 - \gamma\lambda\mu^2\beta\ (\beta-1) = 0,\\ \frac{\varepsilon\lambda^{p+1}}{p+1} + \gamma\lambda\mu^2\beta\ (\beta-1) = 0.\end{gathered} \tag{15}$$

Now we use Mathematica to solve this system of algebraic equations to obtain

$$\begin{gathered}\beta = -\frac{2}{p},\\ \mu = \pm\frac{ip\sqrt{c}}{2\sqrt{\gamma}},\\ \lambda = 2^{-\frac{1}{p}}\left(\frac{c(2+3p+p^2)}{\varepsilon}\right)^{\frac{1}{p}},\end{gathered} \tag{16}$$

substituting the equation (16) in to equation (5), we get

$$u(x,t) = 2^{-\frac{1}{p}}(\frac{c(2+3p+p^2)}{\varepsilon})\ cos^{-\frac{2}{p}}(\pm\frac{ip\sqrt{c}}{2\sqrt{\gamma}}(x-ct)) \tag{17}$$

which is the exact soliton solution of GKDV.

Example 3 Finally, we consider the general equal width wave equation (GEWE).

$$u_t + \varepsilon u^p u_x - v u_{xt} = 0. \tag{18}$$

By using the wave variable $\xi = x - ct$ and $u(x,t) = f(\xi)$, the GEWE is transformed in to

$$-c\frac{df(\xi)}{d\xi} + \varepsilon f^p(\xi)\frac{df(\xi)}{d\xi} + cv\frac{d^3f(\xi)}{d\xi^3} = 0. \tag{19}$$

Since all the terms contain derivatives, integrating once equation (19) gives,

$$-cf(\xi) + \frac{\varepsilon}{p+1}(f(\xi))^{p+1} + cv\frac{d^2f(\xi)}{d\xi^2} = 0. \tag{20}$$

Substituting equation (6) in to equation (20), we get

$$\begin{aligned} &- c\lambda\cos^{\beta}(\mu\xi) + \frac{\varepsilon}{p+1}(\lambda^{p+1}\cos^{\beta(p+1)}(\mu\xi)) - cv\lambda\beta\mu^2\cos^{\beta}(\mu\xi) \\ &+ cv\lambda\mu^2\beta\ (\beta-1)\cos^{\beta-2}\ (\mu\xi) - cv\lambda\mu^2\beta\ (\beta-1)\cos^{\beta}\ (\mu\xi) = 0. \end{aligned} \tag{21}$$

Equating the exponents of cosine functions and the coefficients,

$$\begin{aligned} (p+1)\beta &= \beta - 2, \\ -c\lambda - \gamma\lambda\beta\mu^2 - \gamma\lambda\mu^2\beta\ (\beta-1) &= 0, \\ \frac{\varepsilon\lambda^{p+1}}{p+1} + \gamma\lambda\mu^2\beta\ (\beta-1) &= 0. \end{aligned} \tag{22}$$

(22)is a sistem of algebraic equations.
Using mathematica package for symbolic calculation, the system (22) is solved to obtain

$$\begin{aligned} \beta &= -\frac{2}{p}, \\ \mu &= \pm\frac{ip\sqrt{c}}{2\sqrt{\gamma}}, \\ 2^{-\frac{1}{p}}&\left(\frac{c(2+3p+p^2)}{\varepsilon}\right)^{\frac{1}{p}}. \end{aligned} \tag{23}$$

Finally we substitute equation (22) in to equation (5) and thereby obtain

$$u(x,t) = 2^{-\frac{1}{p}}\left(\frac{c(2+3p+p^2)}{\varepsilon}\right)\ cos^{-\frac{2}{p}}\left(\pm\frac{ip\sqrt{c}}{2\sqrt{\gamma}}(x-ct)\right). \tag{24}$$

Corollary 1 *In this work, the cosine-function method has been successfully applied to find the solution for three nonlinear partial differential equations such as GRLW, GKDV and GEWE equations. The cosine-function method is used to find a new exact solution. Therefore, we can say the proposed method can be exact to solve the problems of nonlinear partial differential equations arising in to the theory of solitons and other areas.*

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