Global Analysis and Discrete Mathematics
Volume 2, Issue 1, pp. 23–31
ISSN: 2476-5341

Solving an Inverse Heat Conduction Problem by Spline Method

Hassan Dana Mazraeh*

Received: 10 July 2016 / Accepted: 1 November 2016

Abstract In this paper, a numerical solution of an inverse non-dimensional heat conduction problem by spline method will be considered. The given heat conduction equation, the boundary condition, and the initial condition are presented in a dimensionless form. A set of temperature measurements at a single sensor location inside the heat conduction body is required. The result show that the proposed method can predict the unknown parameters in the current inverse problem with an acceptable error.

Keywords Inverse heat conduction problem · Spline method · Finite difference method · Tikhonov regularization method

Mathematics Subject Classification (2010) 65M32 · 35K05.

1 Introduction

Solving IHCPs needs additional information about temperature history. This new data is usually given by a temperature sensor which is located on the boundary or inside the body. To date, various methods have been developed for the analysis of the inverse problems and inverse heat conduction problems involving the estimation of heat flux by measuring temperature inside the material [3, 2, 18, 6, 15, 16, 7–9, 5]. In this work, by using spline method, a stable solution for an inverse heat conduction problem will be presented.

*Corresponding author
Hassan Dana Mazraeh
School of Mathematics and Computer Science, Damghan University, P.O. Box 36715-364, Damghan, Iran.
E-mail: dana@du.ac.ir

A possible mathematical model for the temperature in the plate is a one dimensional IHCP, is as follows, [3],

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \qquad 0 < x < 1, \quad 0 < t < t_M, \tag{1a}$$
$$T(x,0) = f(x), \qquad 0 \le x \le 1, \tag{1b}$$
$$T(0,t) = p(t), \qquad 0 \le t \le t_M, \tag{1c}$$
$$T(1,t) = q(t), \qquad 0 \le t \le t_M, \tag{1d}$$

and the overspecified condition

$$T(a,t) = g(t), \qquad 0 \le t \le t_M, \tag{1e}$$

Where $0 < a < 1$ is a fixed point, t_M is a given constant. $f(x)$ is the initial temperature of rod, $p(t)$ is the temperature at the left hand side and $q(t)$ is the temperature of the right hand side. In this context we consider that the functions $f(x)$, $q(t)$ are known functions, while $T(x,t)$ and $p(t)$ are unknown functions which remained. Note that, for an unknown function $p(t)$ we must therefore provide additional information (1e) to provide a unique solution $(T(x,t), p(t))$ to the inverse problem (1).

2 Overview of the Spline method

Consider an inverse diffusion problem described by the equations (1). The application of the present numerical method will fined a solution of problem (1), by using the following step.

2.1 Spline method for discretizing

Let Δ be a partition of the interval $0 \le x \le 1$, which divides $[0,1]$ in to n sub interval with the uniform step length $h = \frac{1}{n}$. Let $k > 0$ be the time direction. the grid points (i,j) are given by $x_i = ih, i = 0(1)n$, $t_j = jk, j = 0, 1, 2, ...$ Let T_i^j be the approximate value of $T(x_i, t_j)$. we next develop an approximation for (1) in which the time derivative is replaced by a finite difference approximation an the space derivative is replaced by the spline approximation [17]. Let n be a positive integer, denote

$$T_x(x_i,t_j) = \frac{\partial T(x_i,t_j)}{\partial x}, T_{xx}(x_i,t_j) = \frac{\partial^2 T(x_i,t_j)}{\partial x^2}, \tag{2}$$

$$S''_{\Delta(x_i,t_j)} = M_i^j + O(h^2), \tag{3}$$

$$T_t(x_i,t_j) = \frac{\partial T(x_i,t_j)}{\partial t}, T_t(x_i,t_j) = \frac{T(x_i,t_{j+1}) - T(x_i,t_j)}{k} + O(k), \tag{4}$$

At the grid point (x_i, t_j), the given differential equation (1) may be discretized as

$$T_t(x_i, t_j) = T_{xx}(x_i, t_j), \tag{5}$$

By using (3), (4) in (5) we get

$$\frac{T(x_i, t_{j+1}) - T(x_i, t_j)}{k} + O(k) = M_i^j + O(h^2), \tag{6}$$

neglecting the truncation error we obtain

$$\frac{T_i^{j+1} - T_i^j}{k} = M_i^j, \tag{7}$$

then we have

$$\frac{T_{i+1}^{j+1} - T_{i+1}^j}{k} = M_{i+1}^j, \tag{8}$$

$$\frac{T_{i-1}^{j+1} - T_{i-1}^j}{k} = M_{i-1}^j. \tag{9}$$

Therefore the following, spline relation, is obtained

$$\alpha M_{i+1}^j + 2\beta M_i^j + \alpha M_{i-1}^j = \frac{1}{h^2}(T_{i+1}^j - 2T_i^j + T_{i-1}^j), \tag{10}$$

where $j = 1(1)n-1$. substituting Eqs. (7)-(9) in to (10), it finally obtained the following schemes

$$\begin{aligned} &\alpha\lambda T_{i+1}^{j+1} + 2\beta\lambda T_i^{j+1} + \alpha\lambda T_{i-1}^{j+1} = \\ &(\alpha\lambda + 1)T_{i+1}^j + (2\beta\lambda - 2)T_i^j + (\alpha\lambda + 1)T_{i-1}^j, \end{aligned} \tag{11}$$

where $\lambda = \frac{h^2}{k}$, $i = 1(1)n-1$, $j = 0, 1, ...,$

$$\alpha = \frac{\theta csc\theta - 1}{\theta^2}, \beta = \frac{1 - \theta cot\theta}{\theta^2}.$$

Equation (11) for $i = 1(1)n-1$ may be written in the following matrix form

$$AT_{j+1} = BT_j + C_j, \tag{12}$$

$$A = \begin{pmatrix} 2\beta\lambda & \alpha\lambda & \dots & 0 & 0 & 0 \\ \alpha\lambda & 2\beta\lambda & \alpha\lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \dots & \alpha\lambda \\ 0 & 0 & \dots & 0 & \alpha\lambda & 2\beta\lambda \end{pmatrix},$$

$$B = \begin{pmatrix} 2\beta\lambda - 2 & \alpha\lambda + 1 & \dots & 0 & 0 & 0 \\ \alpha\lambda + 1 & 2\beta\lambda - 2 & \alpha\lambda + 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \dots & \alpha\lambda + 1 \\ 0 & 0 & \dots & 0 & \alpha\lambda + 1 & 2\beta\lambda - 2 \end{pmatrix},$$

and

$$T_{j+1} = (T_1^{j+1}, T_2^{j+1}, ..., T_{n-1}^{j+1}),$$

$$T_j = (T_1^j, T_2^j, ..., T_{n-1}^j),$$

$$C_j = ((\alpha\lambda + 1)p(jk) - \alpha\lambda p((j+1)k), 0, ..., 0, (\alpha\lambda + 1)q(jk) - \alpha\lambda q((j+1)k)),$$

By choosing suitable values of parameters α, β we obtain various methods for solution of inverse heat conduction problem (1).

Remark 1 In this work the polynomial from proposed for the unknown $p(t)$ before performing the calculation. Therefore $p(t)$ approximate as

$$p(t) = a_0 + a_1 t + a_2 t^2 + ... + a_\jmath t^\jmath, \tag{13}$$

where $a_0, a_1, ..., a_\jmath$ are constants which remain to be determined.

2.2 Least-squares minimization technique

The estimated coefficients $a_s, s = 1, 2, ..., \jmath$ can be determined by using least square method when the sum of the squares of the deviation between the calculated $T_{\frac{a}{h}}^{j+1}$ and the measured $g((j+1)k)$ at $x = a$ is less than a small number. The error in the estimates $E(a_0, a_1, ..., a_\jmath)$ can be expressed as

$$E(a_0, a_1, ..., a_\jmath) = \sum_{j=0}^{i} (T_{\frac{a}{h}}^{j+1} - g((j+1)k))^2, i = 1, 2, ...,$$

which is to be minimized for each interval $t_{m-1} \leq t \leq t_m$, $m = 1, 2, ..., M$. To obtain the minimum value of $E(a_0, a_1, ..., a_\jmath)$, with respect to $a_0, a_1, ..., a_\jmath$, differentiation of $E(a_0, a_1, ..., a_\jmath)$, with respect to $a_0, a_1, ..., a_\jmath$, will be performed [14]. Thus the linear system corresponding to the values of a_s can be expressed as

$$\Lambda\Theta = \Upsilon, \Theta = (a_0, a_1, ..., a_\jmath), \tag{14}$$

The Matrix Λ is ill-conditioned. On the other hand, as g is affected by measurement errors, the estimate of $p(t)$ by (14) will be unstable. Therefore, the Tikhonov regularization method ([19], [13] and [14]) must be used to control this measurement errors.

3 Numerical Results and Discussion

In this section, we are going to study numerically the inverse problems (1) with the unknown boundary condition. The main aim here is to illustrate the applicability of the present method, described in Sections 2 for solving the inverse problem (1). As expected the inverse problems are ill-posed and therefore it is necessary to investigate the stability of the present methods by giving a test problem.

Remark 2 In an inverse problem there are two sources of error in the estimation; the first source is the unavoidable bias deviation, and the second source of error is the variance due to the amplification of measurement errors, [10].

Therefore, we compare exact and approximate solutions by considering total error S defined by

$$S = \left[\frac{1}{N-1}\sum_{\ell=1}^{N}(\widehat{p}_\ell - p_\ell)^2\right]^{\frac{1}{2}}, \tag{15}$$

where N, p and $\widehat{p}$ are the number of estimated values , the estimated values and the exact values, respectively.

Example 1 In this example we solve the problem (1) with given data,

$$T(x,0) = 2(\sin(2x) + \cos(2x)) + \frac{1}{4}x^4,\ 0 \le x \le 1,$$

$$T(1,t) = 2e^{-4t}(sin(2) + cos(2)) + 3t^2 + 3t + 0.25,\quad 0 \le t \le t_M,$$

$$T(0.1,t) = 2e^{-4t}(\sin 0.2 + \cos 0.2) + 3(t^2 + (0.01)t + \frac{0.0001}{12},\ 0 \le t \le t_M.$$

The exact solution of this problem is

$$T(x,t) = 2e^{-4t}(\sin(2x) + \cos(2x)) + 3(t^2 + tx^2 + \frac{1}{12}x^4).$$

and

$$p(t) = 2e^{-4t} + 3t^2, \qquad 0 \le t \le t_M,$$

Our results with Spline method obtained for $(p(t), T(x,t))$ when $t_M = 1$, $k = 0.002$ and $h = 0.1$, $\alpha = \frac{1}{18}$, $\beta = \frac{4}{9}$ by Tikhonov regularization and approximate solution result from Duhamel's [3] method by Tikhonov regularization with noisy data are presented in Tables 1 and 1 and and Figures , , , ;

t t	p(t), Exact	p(t), cubic spline scheme	p(t), Duhamel's scheme
(0.100000)	(1.370640)	(1.371471)	(1.370641)
(0.200000)	(1.018658)	(1.014504)	(1.018660)
(0.300000)	(0.872388)	(0.871520)	(0.872388)
(0.400000)	(0.883793)	(0.881676)	(0.883785)
(0.500000)	(1.020671)	(1.021552)	(1.020665)
(0.600000)	(1.261436)	(1.267904)	(1.261436)
(0.700000)	(1.591620)	(1.597091)	(1.581620)
(0.800000)	(2.001524)	(1.999244)	(2.001424)
(0.900000)	(2.484647)	(2.479174)	(2.487347)
(1.000000)	(3.036631)	(3.040551)	(3.035531)
S		$1.2e-003$	$6.2e-002$

Table 1. The comparison between exact and cubic spline solution and Duhamel's scheme of $p(t)$ with noisy data.

t	Exact $T(0.5,t)$	cubic spline scheme $T(0.5,t)$	Duhamel's scheme $T(0.5,t)$
(0.100000)	(1.973086)	(1.979823)	(1.946453)
(0.200000)	(1.527367)	(1.527793)	(1.499955)
(0.300000)	(1.342989)	(1.342851)	(1.328802)
(0.400000)	(1.353575)	(1.350897)	(1.349223)
(0.500000)	(1.514630)	(1.513900)	(1.516022)
(0.600000)	(1.796328)	(1.795653)	(1.801043)
(0.700000)	(2.178676)	(2.170030)	(2.184406)
(0.800000)	(2.648273)	(2.649084)	(2.655631)
(0.900000)	(3.196135)	(3.195401)	(3.203425)
(1.000000)	(3.816241)	(3.815907)	(3.825544)
S		$1.6e-003$	$1.5e-002$

Table 2. The comparison between exact and cubic spline solution and Duhamel's scheme of $T(x,t)$ with noisy data.

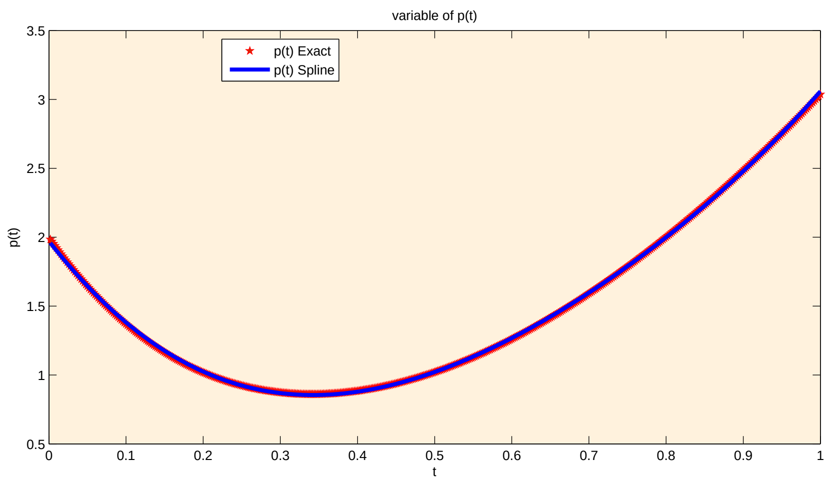

Figure 1. Comparison between the exact results of $p(t)$ and the present numerical results of example 4.1 with noisy data.

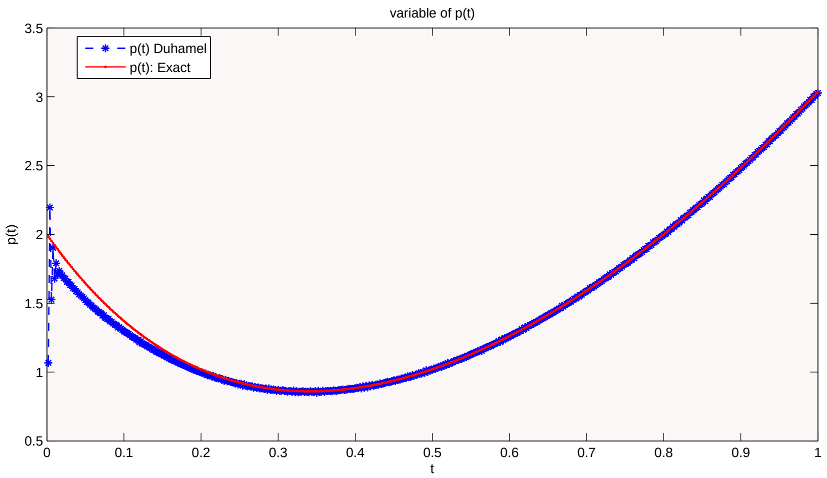

Figure 2. Comparison between the exact results of p(t) and the present numerical results of example 4.1 with noisy data.

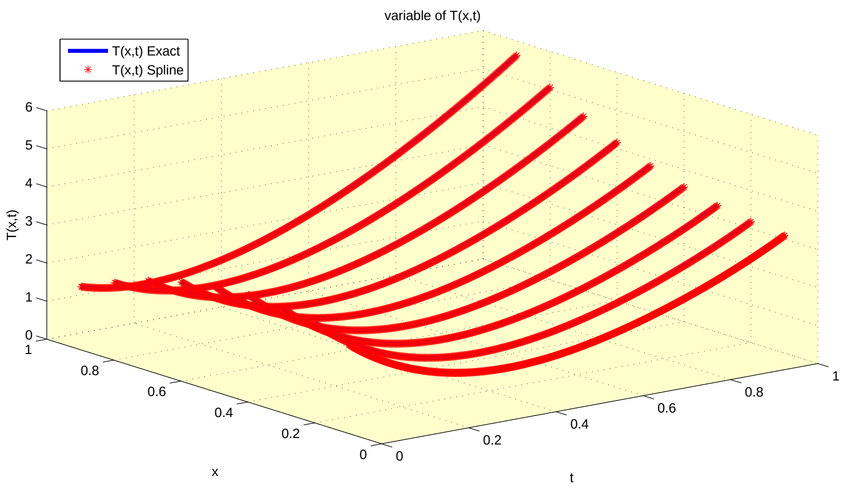

Figure 3. Comparison between the exact results of T(x,t) and the present numerical results of example 4.1 with noisy data.

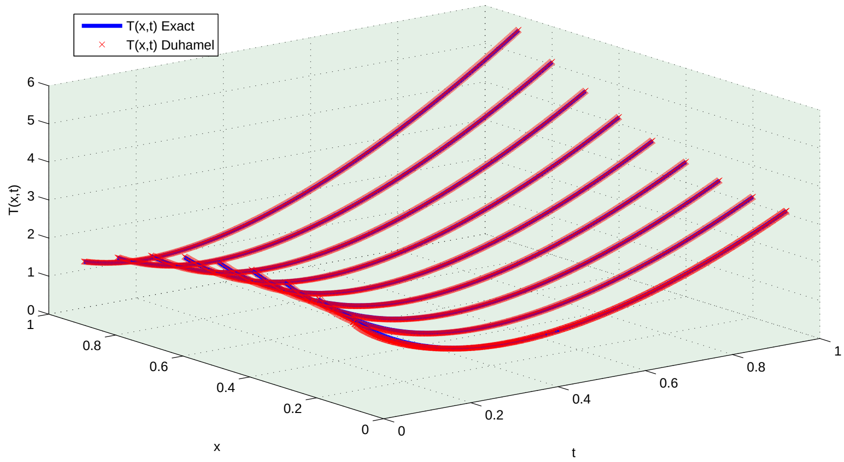

Figure 4. Comparison between the exact results of T(x,t) and the present numerical results of example 4.1 with noisy data.

4 Conclusion

A numerical method to estimate unknown boundary condition is proposed for these kinds of IHCPs and the following results are obtained:

1. The present study successfully applies the numerical method to IHCPs.
2. Numerical results show that an excellent estimation can be obtained within a couple of minutes CPU time at pentium(R) 4 CPU 3.20 GHz.
3. The present method has been found stable with respect to small perturbation in the input data.

References

1. O. M. Alifanov, Inverse Heat Transfer Problems, Springer, NewYork, 1994.
2. N. Al-Khalidy, Analysis of bundary inverse heat conduction problems using space marching with Savitzky-Gollay digital filter, Int. Commun. Heat Mass Transfer 26 (2) (1999) 199-208.
3. J. V. Beck, B. Blackwell, C.R. St. Clair, Inverse Heat Conduction Ill-posed Problems, Wiley, New York, 1985, PP. 1-8.
4. Beck J. V. and Murio D. C., Combined function specification-regularization procedure for solution of inverse heat condition problem, AIAA J. 24 (1986) 180–185.
5. Jianhua Zhou, Yuwen Zhang, J. K. Chen, and Z. C. Feng, Inverse Heat Conduction in a Composite Slab With Pyrolysis Effect and Temperature-Dependent Thermophysical Properties, J. Heat Transfer, 132 (3) (2010) 034502 (3 pages) .
6. T.C. Chen, P.C. Tuan, Input estimation method including finite-element scheme for solving inverse heat conduction problems, Num. Heat Transfer, Part B: Fundamentals 47 (3) (2005) 277-290.
7. D. Lesnic, L.Elliott, D.B. Ingham, Application of the boundary element method to inverse heat conduction problems, Int. J. Heat Mass Transfer 39 (7) (1996) 1503-1517.

8. J. Krejsa, K.A. Woodbury, J.D. Ratliff, M. Raudensky, Assessment of strategies and potential for neural networks in the inverse heat conduction problems, Inverse Probl. Eng. 7 (1999) 197-213.
9. M. Raudensky, J. Horsky, J. Krejsa, Usage of neural network for coupled parameter and function spesification inverse heat conduction problem, Int. Commun. Heat Mass Transfer 22 (5) (1995) 661-670.
10. J.M.G Cabeza, J.A.M Garcia, and A.C. Rodriguez, A Sequential Algorithm of Inverse Heat Conduction Problems Using Singular Value Decomposition, International Journal of Thermal Sciences 44 (2005) 235–244.
11. L. Elden, A Note on the Computation of the Generalized Cross-validation Function for Ill-conditioned Least Squares Problems, BIT, 24 (1984) 467–472.
12. G. H. Golub, M. Heath and G.Wahba, Generalized Cross-validation as a Method for Choosing a Good Ridge Parameter, Technometrics, 21 (2) (1979) 215–223.
13. P.C. Hansen, Analysis of discrete ill-posed problems by means of the L-curve, SIAM Rev 34 (1992) 561–80.
14. C. L. Lawson and R. J. Hanson , Solving Least Squares Problems, Philadelphia, PA: SIAM, (1995).
15. R. Pourgholi and M. Rostamian, A numerical technique for solving IHCPs using Tikhonov regularization method, Applied Mathematical Modelling 34 (8) (2010) 2102–2110.
16. R. Pourgholi, N. Tavallaie and S. Foadian, Applications of Haar basis method for solving some ill-posed inverse problems, J. Math. Chem., 2012, Volume 50, Number 8, Pages 2317-2337, DOI 10.1007/s10910-012-0036-4.
17. J. Rashidinia, R. Jalilian, V. Kazemi, Spline methods for the solutions of hyperbolic equations, Appl. Math. Comput. 190 (2007) 882-886 .
18. M. S. Shin, J. W. Lee, Prediction of the inner wall shape of an eroded furnace by the nonlinear inverse heat conduction technique, JSME Int. J. B 43 (4) (2000) 544-545.
19. A.N. Tikhonov and V.Y. Arsenin, On the solution of ill-posed problems, New York, Wiley, 1977.
20. G. Wahba, Spline Models for Observational Data, CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 59, SIAM, Philadelphia, 1990.