Global Analysis and Discrete Mathematics
Volume 2, Issue 2, pp. 115–134
ISSN: 2476-5341

A New Optimal Solution Concept for Fuzzy Optimal Control Problems

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Received: 7 January 2016 / Accepted: 20 June 2017

Abstract In this paper, we propose the new concept of optimal solution for fuzzy variational problems based on the possibility and necessity measures. Inspired by the well–known embedding theorem, we can transform the fuzzy variational problem into a bi–objective variational problem. Then the optimal solutions of fuzzy variational problem can be obtained by solving its corresponding biobjective variational problem.

Keywords Fuzzy variational problems · Fuzzy Euler–Lagrange conditions · Pareto optimality conditions

Mathematics Subject Classification (2010) 93C42 · 34N05 · 93D05

1 Introduction

The field of calculus of variations is of significant importance in various disciplines such as biology, engineering, signal processing, system identification, control theory, finance and fractional dynamics [4, 6, 12, 1, 18] Functional minimization problems naturally occur in engineering and science where minimization of functionals, such as, Lagrangian, strain, potential, and total energy, etc. give the laws governing the systems behavior. Uncertainty is inherent in most

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real-world systems and fuzziness is a kind of uncertainty in real word problems. The fuzzy calculus of variations forms a suitable setting for mathematical modeling of real–world problems in which uncertainties or vagueness pervade. The fuzzy calculus of variations extends the classical variational calculus considering variables and their derivatives in fuzzy form. There are few papers dealing with fuzzy variational problem [11, 10, 9, 8]. Recently, Farhadinia [11] studied necessary optimality conditions for fuzzy variational problems using the fuzzy differentiability concept due to Buckley and Feuring [5]. Farhadinia's work was generalized by Fard et al. [10, 9, 8]. Fard and Zadeh [10], using α–differentiability concept, obtained an extended fuzzy Euler–Lagrange condition. Fard and Salehi [9] investigate fuzzy fractional Euler–Lagrange equations for fuzzy fractional variational problems defined via generalized fuzzy fractional Caputo type derivatives.

Puri and Ralescu [20] and Kaleva [16] have proven that the set of all fuzzy numbers can be embedded into a Banach space isometrically and isomorphically. Wu and Ma [23] provide a specific Banach space, which shows that the set of all fuzzy numbers can be embedded into the Banach space $\overline{C}[0,1] \times \overline{C}[0,1]$, where $\overline{C}[0,1]$ is the set of all real–valued bounded functions f on $[0,1]$ such that f is left-continuous for any $x \in (0,1]$ and right-continuous at 0, and f has a right limit for any $x \in [0,1)$. Inspired by this specific Banach space, we can transform the fuzzy variational problem into a biobjective variational problem using this embedding theorem. Then necessary and sufficient Pareto optimality conditions are obtained by converting a biobjective variational problem into a single or a family of single variational problems with an auxiliary scalar functional, possibly depending on a parameter.

Due to [17], we will introduce definitions for higher–order fuzzy derivatives and for the sake of convenience, we will obtain the Euler–Lagrange equations to the fuzzy variational problems containing second order derivatives, for the first time in the literature.

In Section 2 and 3, we present some notations on the fuzzy numbers space, differentiability and integrability of a fuzzy mapping and provide the embedding theorem. The main results concerning the optimal solution of fuzzy constraint and unconstraint problems are established in sections 4 and numerical examples are illustrated for providing the basic techniques to compute the optimal solutions by resorting to Euler–Lagrange equation, and finally, conclusions are discussed in Section 5.

2 Preliminaries

Let us denote by $\mathbb{R}_f$ the class of fuzzy numbers, i.e., normal, convex, upper semicontinuous and compactly supported fuzzy subsets of the real numbers. For $0 < r \leq 1$, let $[\tilde{u}]^r = \{x \in \mathbb{R}; \tilde{u}(x) \geq r\}$ and $[\tilde{u}]^0 = \overline{\{x \in \mathbb{R}; \tilde{u}(x) \geq 0\}}$. Then, it is well known that $[\tilde{u}]^r$ is a bounded closed interval for any $r \in [0,1]$. The notation $[\tilde{u}]^r = [\underline{u}^r, \overline{u}^r]$ denotes explicitly the r-level set of $\tilde{u}$.

The following remark shows when $[\underline{u}^r, \overline{u}^r]$ is a valid r-level set of a fuzzy number.

Remark 1 (See [17]) The sufficient and necessary conditions for $[\underline{u}^r, \overline{u}^r]$ to define the parametric form of a fuzzy number are as follows:

(i) $\underline{u}^r$ is a bounded monotonic increasing (nondecreasing) left-continuous function $\forall r \in (0,1]$ and right-continuous for $r = 0$.
(ii) $\overline{u}^r$ is a bounded monotonic decreasing (nonincreasing) left-continuous function $\forall r \in (0,1]$ and right-continuous for $r = 0$.
(iii) $\underline{u}^r \leq \overline{u}^r,\ 0 \leq r \leq 1$.

For $\tilde{u}, \tilde{v} \in \mathbb{R}_f$ and $\lambda \in \mathbb{R}$, the sum $\tilde{u}+\tilde{v}$ and the product $\lambda \cdot \tilde{u}$ are defined by $[\tilde{u}+\tilde{v}]^r = [\tilde{u}]^r + [\tilde{v}]^r$ and $[\lambda \cdot \tilde{u}]^r = \lambda[\tilde{u}]^r$ for all $r \in [0,1]$, where $[\tilde{u}]^r + [\tilde{v}]^r$ means the usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda[\tilde{u}]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$. The product $\tilde{u}\odot\tilde{v}$ of fuzzy numbers $\tilde{u}$ and $\tilde{v}$, is defined by $[\tilde{u}\odot\tilde{v}]^r = [\min\{\underline{u}^r\underline{v}^r, \underline{u}^r\overline{v}^r, \overline{u}^r\underline{v}^r, \overline{u}^r\overline{v}^r\}, \max\{\underline{u}^r\underline{v}^r, \underline{u}^r\overline{v}^r, \overline{u}^r\underline{v}^r, \overline{u}^r\overline{v}^r\}]$.

We say that the fuzzy number $\tilde{u}$ is triangular if $\underline{u}^1 = \overline{u}^1$, $\underline{u}^r = \underline{u}^1 - (1 - r)(\underline{u}^1 - \underline{u}^0)$ and $\overline{u}^r = \underline{u}^1 - (1-r)(\overline{u}^0 - \underline{u}^1)$. The triangular fuzzy number u is generally denoted by $\tilde{u} =< \underline{u}^0, \underline{u}^1, \overline{u}^0 >$. We define the fuzzy zero $\tilde{0}_x$ as

$$\tilde{0}_x = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

Definition 1 (See [11]) We say that $\tilde{f} : [a,b] \to \mathbb{R}_f$ is continuous at $x \in [a,b]$, if both $\underline{f}^r(x)$ and $\overline{f}^r(x)$ are continuous functions of $x \in [a,b]$ for all $r \in [0,1]$.

Definition 2 (See [3]) The generalized Hukuhara difference of two fuzzy numbers $\tilde{x}, \tilde{y} \in \mathbb{R}_f$ (gH-difference for short) is defined as follows:

$$\tilde{x} \ominus_{gH} \tilde{y} = \tilde{z} \Leftrightarrow \tilde{x} = \tilde{y} + \tilde{z} \text{ or } \tilde{y} = \tilde{x} + (-1)\tilde{z}.$$

If $\tilde{z} = \tilde{x} \ominus_{gH} \tilde{y}$ exists as a fuzzy number, then its level cuts $[\underline{z}^r, \overline{z}^r]$ are obtained by $\underline{z}^r = \min\{\underline{x}^r - \underline{y}^r, \overline{x}^r - \overline{y}^r\}$ and $\overline{z}^r = \max\{\underline{x}^r - \underline{y}^r, \overline{x}^r - \overline{y}^r\}$ for all $r \in [0,1]$.

Definition 3 (See [15]) Let $x \in (a,b)$ and h be such that $x+h \in (a,b)$. The generalized Hukuhara derivative of a fuzzy-valued function $\tilde{f} : (a,b) \to \mathbb{R}_f$ at x is defined by

$$\mathcal{D}^{(1)}_{gH}\tilde{f}(x) = \lim_{h\to 0} \frac{\tilde{f}(x+h) \ominus_{gH} \tilde{f}(x)}{h}. \tag{1}$$

If $\mathcal{D}^{(1)}_{gH}\tilde{f}(x) \in \mathbb{R}_f$ satisfying (1) exists, then we say that $\tilde{f}$ is generalized Hukuhara differentiable (gH-differentiable for short) at x. Also, we say that $\tilde{f}$ is $[(1)-gH-]$differentiable at x (denoted by $\mathcal{D}^{(1)}_1\tilde{f}$) if $[\mathcal{D}^{(1)}_{gH}\tilde{f}(x)]^r = [\dot{\underline{f}}^r(x), \dot{\overline{f}}^r(x)]$, and that $\tilde{f}$ is $[(2)-gH]-$differentiable at x (denoted by $\mathcal{D}^{(1)}_2\tilde{f}$) if $[\mathcal{D}^{(1)}_{gH}\tilde{f}(x)]^r = [\dot{\overline{f}}^r(x), \dot{\underline{f}}^r(x)]$, $r \in [0,1]$.

Due to [17], we introduce definitions for higher-order derivatives based on the selection of derivative type in each step of differentiation. For the sake of convenience, we concentrate on the second-order case.

For a given fuzzy function $\tilde{f}$, we have two possibilities to obtain the derivative of $\tilde{f}$ on x: $\mathcal{D}_1^{(1)} f(x)$, $\mathcal{D}_2^{(1)} f(x)$. Then for each of these two derivatives, we have again two possibilities:

$$\mathcal{D}_1^{(1)}(\mathcal{D}_1^{(1)}\tilde{f}(x)),\ \mathcal{D}_2^{(1)}(\mathcal{D}_1^{(1)}\tilde{f}(x))$$

and

$$\mathcal{D}_1^{(1)}(\mathcal{D}_2^{(1)}\tilde{f}(x)),\ \mathcal{D}_2^{(1)}(\mathcal{D}_2^{(1)}\tilde{f}(x)),$$

respectively.

Definition 4 (See [17]) Let $\tilde{f} : (a,b) \longrightarrow \mathbb{R}_f$ and $n,m = 1,2$. We say that $\tilde{f}$ is $[(n,m) - gH]-$differentiable at $x_0 \in (a,b)$ if $\mathcal{D}_n^{(1)}\tilde{f}(x)$ exists on a neighborhood of x_0 as a fuzzy function and it is $[(m) - gH]-$differentiable at x_0. The second derivatives of $\tilde{f}$ are denoted by $\mathcal{D}_{(n,m)}^{(2)}\tilde{f}(x)$ for $n,m = 1,2$.

Theorem 1 (See [17]) *Let $\mathcal{D}_1^{(1)}\tilde{f} : (a,b) \longrightarrow \mathbb{R}_f$ or $\mathcal{D}_2^{(1)}\tilde{f} : (a,b) \longrightarrow \mathbb{R}_f$ be fuzzy functions,*

(i) If $\mathcal{D}_1^{(1)}\tilde{f}$ is $[(1)-gH]-$differentiable, then $\dot{\underline{f}}^r$ and $\dot{\overline{f}}^r$ are differentiable functions and $[\mathcal{D}_{(1,1)}^{(2)}\tilde{f}(x)]^r = [\ddot{\underline{f}}^r, \ddot{\overline{f}}^r]$ and we say that $\tilde{f}$ is $[(1,1)-gH]-$differentiable at x.

(ii) If $\mathcal{D}_1^{(1)}\tilde{f}$ is $[(2)-gH]-$differentiable, then $\dot{\underline{f}}^r$ and $\dot{\overline{f}}^r$ are differentiable functions and $[\mathcal{D}_{(1,2)}^{(2)}\tilde{f}(x)]^r = [\ddot{\overline{f}}^r, \ddot{\underline{f}}^r]$ and we say that $\tilde{f}$ is $[(1,2)-gH]-$differentiable at x.

(iii) If $\mathcal{D}_2^{(1)}\tilde{f}$ is $[(1)-gH]-$differentiable, then $\dot{\underline{f}}^r$ and $\dot{\overline{f}}^r$ are differentiable functions and $[\mathcal{D}_{(2,1)}^{(2)}\tilde{f}(x)]^r = [\ddot{\overline{f}}^r, \ddot{\underline{f}}^r]$ and we say $\tilde{f}$ is $[(2,1)-gH]-$differentiable at x.

(iv) If $\mathcal{D}_2^{(1)}\tilde{f}$ is $[(2)-gH]-$differentiable, then $\dot{\underline{f}}^r$ and $\dot{\overline{f}}^r$ are differentiable functions and $[\mathcal{D}_{(2,2)}^{(2)}\tilde{f}(x)]^r = [\ddot{\underline{f}}^r, \ddot{\overline{f}}^r]$ and we say that $\tilde{f}$ is $[(2,2)-gH]-$differentiable at x.

If the fuzzy function $\tilde{f}(x)$ is continuous, then its definite integral exists. Furthermore,

$$\left(\underline{\int_a^b \tilde{f}(x)dx}\right)^r = \int_a^b \underline{f}^r(x)dx, \quad \left(\overline{\int_a^b \tilde{f}(x)dx}\right)^r = \int_a^b \overline{f}^r(x)dx.$$

Definition 5 (Partial ordering) Let $\tilde{a}, \tilde{b} \in \mathbb{R}_f$. We write $\tilde{a} \preceq \tilde{b}$, if and only if $\underline{a}^r \leq \underline{b}^r$ and $\overline{a}^r \leq \overline{b}^r$ for all $r \in [0,1]$. We also write $\tilde{a} \prec \tilde{b}$, if and only if

$$\begin{cases} \underline{a}^r < \underline{b}^r \\ \overline{a}^r \leq \overline{b}^r \end{cases} \text{or} \quad \begin{cases} \underline{a}^r \leq \underline{b}^r \\ \overline{a}^r < \overline{b}^r \end{cases} \text{or} \quad \begin{cases} \underline{a}^r < \underline{b}^r \\ \overline{a}^r < \overline{b}^r. \end{cases} \tag{2}$$

Moreover, $\tilde{a} \approx \tilde{b}$ if and only if $\tilde{a} \preceq \tilde{b}$ and $\tilde{a} \succeq \tilde{b}$, that is, $[\tilde{a}]^r = [\tilde{b}]^r$ for all $r \in [0,1]$.

3 Embedding theorem

Now we are going to embed $\mathbb{R}_f$ into a Banach space isometrically and isomorphically. Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$. The *Hausdorff metric* is defined by

$$d_H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}||a-b||, \sup_{b\in B}\inf_{a\in A}||a-b||\right\}.$$

According to [20], we define the metric d_f in $\mathbb{R}_f$ as

$$d_f(\tilde{a},\tilde{b}) = \sup_{0\le r\le 1} d_H([\tilde{a}]^r,[\tilde{b}]^r).$$

For $\tilde{a},\tilde{b} \in \mathbb{R}_f$, we have

$$d_H([\tilde{a}]^r,[\tilde{b}]^r) = \max\left\{|\underline{a}^r - \underline{b}^r|, |\overline{a}^r - \overline{b}^r|\right\}.$$

The space $\overline{C}[0,1]$ is the set of all real–valued bounded functions f on $[0,1]$ such that f is left-continuous for any $x \in (0,1]$ and right-continuous at 0 and f has a right limit for any $x \in [0,1)$. Then $(\overline{C}[0,1], ||.||)$ is a Banach space with the norm defined by $||f|| = \sup_{x\in[0,1]}|f(x)|$. Furthermore, $(\overline{C}[0,1] \times \overline{C}[0,1], ||.||)$ is also a Banach space with the norm defined by

$$||(f,g)|| = \max\{||f||, ||g||\},$$

where $(f,g) \in \overline{C}[0,1] \times \overline{C}[0,1]$ (See [23]).

Let $\tilde{a}$ be a fuzzy number, i.e., $\tilde{a} \in \mathbb{R}_f$. We consider $\underline{a}^r$ and $\overline{a}^r$, as the functions of $r \in [0,1]$. Then we can define the embedding function $\pi : \mathbb{R}_f \longrightarrow \overline{C}[0,1]\times\overline{C}[0,1]$ by $\pi(\tilde{a}) = (\underline{a}^r, \overline{a}^r)$. The embedding theorem is presented below.

Theorem 2 (Embedding Theorem) *(See [23]) The function $\pi : \mathbb{R}_f \longrightarrow \overline{C}[0,1] \times \overline{C}[0,1]$ is defined by $\pi(\tilde{a}) = (\underline{a}^r, \overline{a}^r)$. Then the following properties hold true.*

(i) $\pi$is injective.

(ii) $\pi((s.\tilde{a}) + (t.\tilde{b})) = s\pi(\tilde{a}) + t\pi(\tilde{b})$ for all $\tilde{a},\tilde{b} \in \mathbb{R}_f, s \ge 0, t \ge 0$.

(iii)$d_f(\tilde{a},\tilde{b}) = ||\pi(\tilde{a}) - \pi(\tilde{b})||$,

for $r \in [0,1]$. That is to say, $\mathbb{R}_f$ can be embedded into $\overline{C}[0,1] \times \overline{C}[0,1]$ isometrically and isomorphically.

The above theorem says that each element in $\mathbb{R}_f$ can be regarded as an element in $\overline{C}[0,1] \times \overline{C}[0,1]$.

Let f_1, f_2, g_1 and g_2 be real–valued functions defined on the same real vector space V. We write, $(f_1,g_1) \le (f_2,g_2)$ if and only if $f_1(x_0) \le f_2(x_0)$ and $g_1(x_0) \le g_2(x_0)$ for any fixed $x_0 \in V$. We also write $(f_1,g_1) < (f_2,g_2)$ if and only if

$$\begin{cases} f_1(x_0) < f_2(x_0) \\ g_1(x_0) \le g_2(x_0) \end{cases} \text{or} \quad \begin{cases} f_1(x_0) \le f_2(x_0) \\ g_1(x_0) < g_2(x_0) \end{cases} \text{or} \quad \begin{cases} f_1(x_0) < f_2(x_0) \\ g_1(x_0) < g_2(x_0). \end{cases} \tag{3}$$

Moreover we write $(f_1,g_1) = (f_2,g_2)$ if and only if $f_1(x_0) = f_2(x_0)$ and $g_1(x_0) = g_2(x_0)$.

Lemma 1 (Order Preserving) *Let* $\tilde{a}, \tilde{b} \in \mathbb{R}_f$ *and* π *be the embedding function defined in Theorem 2 then* $\tilde{a} \preceq \tilde{b}$ *if and only if* $\pi(\tilde{a}) \leq \pi(\tilde{b})$. *We also have* $\tilde{a} \prec \tilde{b}$ *if and only if* $\pi(\tilde{a}) < \pi(\tilde{b})$. *Moreover* $\tilde{a} \approx \tilde{b}$ *if and only if* $\pi(\tilde{a}) = \pi(\tilde{b})$.

Proof. From Definition 5, we see that $\tilde{a} \preceq \tilde{b}$ if and only if $\underline{a}^r \leq \underline{b}^r$ and $\overline{a}^r \leq \overline{b}^r$. Then $\pi(\tilde{a}) = (\underline{a}^r, \overline{a}^r) \leq \pi(\tilde{b}) = (\underline{b}^r, \overline{b}^r)$ for $r \in [0,1]$. Similarly, from expressions (2) and (3), we see that $\tilde{a} \prec \tilde{b}$ if and only if $\pi(\tilde{a}) < \pi(\tilde{b})$. Since $\tilde{a} \approx \tilde{b}$ if and only if $[\underline{a}^r, \overline{a}^r] = [\underline{b}^r, \overline{b}^r]$, we see that $\tilde{a} \approx \tilde{b}$ if and only if $\pi(\tilde{a}) = \pi(\tilde{b})$. □

4 Optimality for fuzzy isoperimetric problem

The problem involving minimization of a functional while giving a integral constraints is called the isoperimetric problem. Let $\mathbf{y} \in \mathbf{C}^4[a,b]$, where $\mathbf{y} = (y_1, ..., y_n)$ and $y_k \in C^4[a,b]$ for $k = 1, ..., n$, and $\mathbf{y}_a,\ \mathbf{y}_b$ are given in $\mathbb{R}^n$. Then we consider the following classic isoperimetric problem:

$$\begin{gathered} J(\mathbf{y}) = \int_a^b L(x, \mathbf{y}(x), \dot{\mathbf{y}}(x), \ddot{\mathbf{y}}(x))dx \longrightarrow \min, \\ \int_a^b g_i(x, \mathbf{y}(x), \dot{\mathbf{y}}(x), \ddot{\mathbf{y}}(x))dx = C_i, \quad i = 1, ..., m, \\ \mathbf{y}(a) = \mathbf{y}_a, \quad \mathbf{y}(b) = \mathbf{y}_b, \\ \dot{\mathbf{y}}(a) = \dot{\mathbf{y}}_a, \quad \dot{\mathbf{y}}(b) = \dot{\mathbf{y}}_b. \end{gathered} \tag{4}$$

Assume that g_i has continuous partial derivatives of third order with respect to all its arguments for $i = 1, ..., m$. The well–known Euler–lagrange equation for problem (4) is stated as follows.

Theorem 3 (See [22]) *If* $\mathbf{y} \in \mathbf{C}^4[a,b]$ *is an extremal for problem* (4)*, then there exist constants* λ_i *for* $i = 1, ..., m$*, such that*

$$\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{y}_k}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial \ddot{y}_k}\right) = 0, \quad k = 1, .., n,$$

for all $x \in [a,b]$*, where*

$$F = L + \Sigma_{i=1}^m \lambda_i g_i.$$

Definition 6 We say that $\tilde{y} = \tilde{y}(x)$ is admissible, if it satisfies the end–conditions and $\underline{y}^r, \overline{y}^r$ have continuous fourth order derivative. We denote the set of all admissible curves by $\tilde{X}_{ad}$.

Now we consider the following fuzzy isoperimetric problem:

$$\begin{gathered} \tilde{J}(\tilde{y}) \approx \int_a^b \tilde{L}(x, \tilde{y}(x), \dot{\tilde{y}}(x), \ddot{\tilde{y}}(x))dx \longrightarrow \min, \\ \tilde{I}(\tilde{y}) \approx \int_a^b \tilde{g}(x, \tilde{y}(x), \dot{\tilde{y}}(x), \ddot{\tilde{y}}(x))dx \approx \tilde{C}, \\ \tilde{y}(a) \approx \tilde{y}_a, \quad \tilde{y}(b) \approx \tilde{y}_b, \\ \dot{\tilde{y}}(a) \approx \dot{\tilde{y}}_a, \quad \dot{\tilde{y}}(b) \approx \dot{\tilde{y}}_b, \end{gathered} \tag{5}$$

where $\tilde{C} \in \mathbb{R}_f$ is a given fuzzy number and $\underline{L}^r$ and $\overline{L}^r$ have continuous partial derivatives of third order with respect to all its arguments.

We say that $\tilde{y}^*(x)$ is an optimal solution of problem (5), if $\tilde{y}^*(x)$ is an admissible curve and there exists no admissible curve $\tilde{y}(x)(\neq \tilde{y}^*(x))$ of problem (5) such that $\tilde{J}(\tilde{y}) \prec \tilde{J}(\tilde{y}^*)$.

Let π be the function defined in Theorem 2. Then we consider the following optimization problem

$$\begin{aligned}
\pi(\tilde{J}(\tilde{y})) &= \pi\left(\int_a^b \tilde{L}(x, \tilde{y}(x), \dot{\tilde{y}}(x), \ddot{\tilde{y}}(x))dx\right) \longrightarrow \min, \\
\pi(\tilde{I}(\tilde{y})) &= \pi\left(\int_a^b \tilde{g}(x, \tilde{y}(x), \dot{\tilde{y}}(x), \ddot{\tilde{y}}(x))dx\right) = \pi(\tilde{C}), \\
\pi(\tilde{y}(a)) &= \pi(\tilde{y}_a), \quad \pi(\tilde{y}(b)) = \pi(\tilde{y}_b), \\
\pi(\dot{\tilde{y}}(a)) &= \pi(\dot{\tilde{y}}_a), \quad \pi(\dot{\tilde{y}}(b)) = \pi(\dot{\tilde{y}}_b).
\end{aligned} \tag{6}$$

To simplify the writing, we denote

$$([y]^r(x)) = (x, \underline{y}^r(x), \overline{y}^r(x), \underline{\dot{y}}^r(x), \dot{\overline{y}}^r(x), \underline{\ddot{y}}^r(x), \ddot{\overline{y}}^r(x))$$

where $r \in [0,1]$. From the embedding Theorem 2, we have

$$\begin{aligned}
\pi(\tilde{J}(\tilde{y})) &= \left(\underline{J}^r(\underline{y}^r, \overline{y}^r), \overline{J}^r(\underline{y}^r, \overline{y}^r)\right) = \left(\int_a^b \underline{L}^r([y]^r(x))dx, \int_a^b \overline{L}^r([y]^r(x))dx\right), \\
\pi(\tilde{I}(\tilde{y})) &= \left(\underline{I}^r(\underline{y}^r, \overline{y}^r), \overline{I}^r(\underline{y}^r, \overline{y}^r)\right) = \left(\int_a^b \underline{g}^r([y]^r(x))dx, \int_a^b \overline{g}^r([y]^r(x))dx\right) = (\underline{C}^r, \overline{C}^r), \\
\pi(\tilde{y}(a)) &= \pi(\tilde{y}_a) = (\underline{y}^r(a), \overline{y}^r(a)) = (\underline{y}^r_a, \overline{y}^r_a), \\
\pi(\tilde{y}(b)) &= \pi(\tilde{y}_b) = (\underline{y}^r(b), \overline{y}^r(b)) = (\underline{y}^r_b, \overline{y}^r_b), \\
\pi(\dot{\tilde{y}}(a)) &= \pi(\dot{\tilde{y}}_a) = (\underline{\dot{y}}^r(a), \dot{\overline{y}}^r(a)) = (\underline{\dot{y}}^r_a, \dot{\overline{y}}^r_a), \\
\pi(\dot{\tilde{y}}(b)) &= \pi(\dot{\tilde{y}}_b) = (\underline{\dot{y}}^r(b), \dot{\overline{y}}^r(b)) = (\underline{\dot{y}}^r_b, \dot{\overline{y}}^r_b).
\end{aligned} \tag{7}$$

We say that $\tilde{y}^*$ is an optimal solution of problem (6) if there exists no admissible curve $\tilde{y}(\neq \tilde{y}^*)$ such that $\pi(\tilde{J}(\tilde{y})) < \pi(\tilde{J}(\tilde{y}^*))$. In other words, $\tilde{y}^*$ is a solution of problem (6) if there exists no $\tilde{y}(\neq \tilde{y}^*)$ such that

$$\begin{cases} \underline{J}^r(\underline{y}^r, \overline{y}^r) < \underline{J}^r(\underline{y}^{*r}, \overline{y}^{*r}) \\ \overline{J}^r(\underline{y}^r, \overline{y}^r) \le \overline{J}^r(\underline{y}^{*r}, \overline{y}^{*r}) \end{cases} \quad \text{or} \quad \begin{cases} \underline{J}^r(\underline{y}^r, \overline{y}^r) \le \underline{J}^r(\underline{y}^{*r}, \overline{y}^{*r}) \\ \overline{J}^r(\underline{y}^r, \overline{y}^r) < \overline{J}^r(\underline{y}^{*r}, \overline{y}^{*r}) \end{cases}$$

$$\text{or} \quad \begin{cases} \underline{J}^r(\underline{y}^r, \overline{y}^r) < \underline{J}^r(\underline{y}^{*r}, \overline{y}^{*r}) \\ \overline{J}^r(\underline{y}^r, \overline{y}^r) < \overline{J}^r(\underline{y}^{*r}, \overline{y}^{*r}) \end{cases}$$

for $r \in [0,1]$. Inspired by Equation (7), we consider the following biobjective variational problem

$$
\begin{aligned}
\Big(\underline{J}^r(\underline{y}^r,\overline{y}^r),\overline{J}^r(\underline{y}^r,\overline{y}^r)\Big) &= \left(\int_a^b \underline{L}^r([y]^r(x)),\int_a^b \overline{L}^r([y]^r(x))\right) \to \min,\\
\underline{I}^r(\underline{y}^r,\overline{y}^r) &= \int_a^b \underline{g}^r([y]^r(x)) = \underline{C}^r,\\
\overline{I}^r(\underline{y}^r,\overline{y}^r) &= \int_a^b \overline{g}^r([y]^r(x)) = \overline{C}^r,\\
\underline{y}^r(a) = \underline{y}^r_a, &\quad \overline{y}^r(a) = \overline{y}^r_a,\\
\underline{y}^r(b) = \underline{y}^r_b, &\quad \overline{y}^r(b) = \overline{y}^r_b,\\
\dot{\underline{y}}^r(a) = \dot{\underline{y}}^r_a, &\quad \dot{\overline{y}}^r(a) = \dot{\overline{y}}^r_a,\\
\dot{\underline{y}}^r(b) = \dot{\underline{y}}^r_b, &\quad \dot{\overline{y}}^r(b) = \dot{\overline{y}}^r_b.
\end{aligned}
\tag{8}
$$

Definition 7 Let function $(\underline{y}^{*r},\overline{y}^{*r})$ satisfies the conditions of problem (8). $(\underline{y}^{*r},\overline{y}^{*r})$ is called a Pareto optimal solution to problem (8) if does not exist $(\underline{y}^r,\overline{y}^r)$ for problem (8) with

$$\forall i \in \{1,2\} : J^i(\underline{y}^r,\overline{y}^r) \leq J^i(\underline{y}^{*r},\overline{y}^{*r}) \cap \exists i \in \{1,2\} : J^i(\underline{y}^r,\overline{y}^r) < J^i(\underline{y}^{*r},\overline{y}^{*r})$$

where $J^1 = \underline{J}^r; J^2 = \overline{J}^r$ and $r \in [0,1]$.

In other words, $(\underline{y}^{*r},\overline{y}^{*r})$ is a Pareto optimal solution to problem (8) if there exists no $(\underline{y}^r,\overline{y}^r)(\neq (\underline{y}^{*r},\overline{y}^{*r}))$ such that

$$
\begin{cases}
\underline{J}^r(\underline{y}^r,\overline{y}^r) < \underline{J}^r(\underline{y}^{*r},\overline{y}^{*r})\\
\overline{J}^r(\underline{y}^r,\overline{y}^r) \leq \overline{J}^r(\underline{y}^{*r},\overline{y}^{*r})
\end{cases}
\quad \text{or} \quad
\begin{cases}
\underline{J}^r(\underline{y}^r,\overline{y}^r) \leq \underline{J}^r(\underline{y}^{*r},\overline{y}^{*r})\\
\overline{J}^r(\underline{y}^r,\overline{y}^r) < \overline{J}^r(\underline{y}^{*r},\overline{y}^{*r})
\end{cases}
$$

$$
\text{or} \quad
\begin{cases}
\underline{J}^r(\underline{y}^r,\overline{y}^r) < \underline{J}^r(\underline{y}^{*r},\overline{y}^{*r})\\
\overline{J}^r(\underline{y}^r,\overline{y}^r) < \overline{J}^r(\underline{y}^{*r},\overline{y}^{*r})
\end{cases}
$$

for $r \in [0,1]$.

For enhanced notions of Pareto optimality of constrained multiobjective problems, the reader is referred to [2].

Lemma 2 *Problems* (5) *and* (6) *have the identical feasible sets.*

Proof. From Lemma 1,

$$
\begin{aligned}
\int_a^b \tilde{g}\left(x,\tilde{y}(x),\dot{\tilde{y}}(x),\ddot{\tilde{y}}(x)\right)dx &\approx \tilde{C},\\
\tilde{y}(a) \approx \tilde{y}_a, &\quad \tilde{y}(b) \approx \tilde{y}_b,\\
\dot{\tilde{y}}(a) \approx \dot{\tilde{y}}_a, &\quad \dot{\tilde{y}}(b) \approx \dot{\tilde{y}}_b,
\end{aligned}
$$

if and only if

$$\pi\left(\int_a^b \tilde{g}\left(x, \tilde{y}(x), \dot{\tilde{y}}(x), \ddot{\tilde{y}}(x)\right) dx\right) = \pi(\tilde{C}),$$
$$\pi(\tilde{y}(a)) = \pi(\tilde{y}_a), \quad \pi(\tilde{y}(b)) = \pi(\tilde{y}_b),$$
$$\pi(\dot{\tilde{y}}(a)) = \pi(\dot{\tilde{y}}_a), \quad \pi(\dot{\tilde{y}}(b)) = \pi(\dot{\tilde{y}}_b).$$

This completes the proof. □

Lemma 3 *$\tilde{y}^*$ is an optimal solution of problem* (5) *if and only if $\tilde{y}^*$ is an optimal solution of problem* (6).

Proof. From Lemma 1, $\tilde{J}(\tilde{y}) \prec \tilde{J}(\tilde{y}^*)$ if and only if $\pi(\tilde{J}(\tilde{y})) < \pi(\tilde{J}(\tilde{y}^*))$ and from Lemma 2, problems (5) and (6) have the identical feasible sets. This completes the proof. □

Lemma 4 *If $(\underline{y}^{*r}, \overline{y}^{*r})$ is a Pareto optimal solution of the variational problem* (8) *then $\tilde{y}^*$ is an optimal solution of problem* (6) *where $[\tilde{y}^*]^r = (\underline{y}^{*r}, \overline{y}^{*r})$.*

Proof. Since $(\underline{y}^{*r}, \overline{y}^{*r})$ is a feasible solution of problem (8), we have

$$\int_a^b \underline{g}^r([y^*]^r(x))dx = \underline{C}^r, \quad \int_a^b \overline{g}^r([y^*]^r(x))dx = \overline{C}^r,$$
$$\underline{y}^{*r}(a) = \underline{y}_a^{*r}, \quad \overline{y}^{*r}(a) = \overline{y}_a^{*r},$$
$$\underline{y}^{*r}(b) = \underline{y}_b^{*r}, \quad \overline{y}^{*r}(b) = \overline{y}_b^{*r},$$
$$\underline{\dot{y}}^{*r}(a) = \underline{\dot{y}}_a^{*r}, \quad \dot{\overline{y}}^{*r}(a) = \dot{\overline{y}}_a^{*r},$$
$$\underline{\dot{y}}^{*r}(b) = \underline{\dot{y}}_b^{*r}, \quad \dot{\overline{y}}^{*r}(b) = \dot{\overline{y}}_b^{*r},$$

From Eqs (7), $\tilde{y}^*$ is a feasible solution of problem (6). Suppose that $\tilde{y}^*$ is not an optimal solution of problem (6). Then there exists a feasible solution $\tilde{y}(\neq \tilde{y}^*)$ of problem (6) such that

$$\begin{cases} \underline{J}^r(\underline{y}^r, \overline{y}^r) \leq \underline{J}^r(\underline{y}^{*r}, \overline{y}^{*r}) \\ \overline{J}^r(\underline{y}^r, \overline{y}^r) < \overline{J}^r(\underline{y}^{*r}, \overline{y}^{*r}) \end{cases} \quad \text{or} \quad \begin{cases} \underline{J}^r(\underline{y}^r, \overline{y}^r) < \underline{J}^r(\underline{y}^{*r}, \overline{y}^{*r}) \\ \overline{J}^r(\underline{y}^r, \overline{y}^r) \leq \overline{J}^r(\underline{y}^{*r}, \overline{y}^{*r}) \end{cases}$$

$$\text{or} \quad \begin{cases} \underline{J}^r(\underline{y}^r, \overline{y}^r) < \underline{J}^r(\underline{y}^{*r}, \overline{y}^{*r}) \\ \overline{J}^r(\underline{y}^r, \overline{y}^r) < \overline{J}^r(\underline{y}^{*r}, \overline{y}^{*r}). \end{cases}$$

Therefore $(\underline{y}^{*r}, \overline{y}^{*r})$ is not a Pareto optimal solution of problem (8). which contradicts the hypothesis. This completes the proof. □

Theorem 4 *If $(\underline{y}^{*r}, \overline{y}^{*r})$ is a Pareto optimal solution of the variational problem* (8) *then $\tilde{y}^*$ is an optimal solution of fuzzy isoperimetric problem* (5) *where $[\tilde{y}^*]^r = (\underline{y}^{*r}, \overline{y}^{*r})$.*

Proof. From the proof of Lemma 4, we see that $\tilde{y}^*$ is a feasible solution of problem (6). Now from Lemma 2, we also see that $\tilde{y}^*$ is a feasible solution of problem (5). Then the result follows from Lemma 3 and 4 immediately. □

We obtain a sufficient condition for Pareto optimality by modifying the biobjective problem (8) into the following weighting problem:

$$\begin{aligned} w\int_a^b \underline{L}^r([y]^r(x))dx + (1-w)\int_a^b \overline{L}^r([y]^r(x))dx \longrightarrow \min, \\ \int_a^b \underline{g}^r([y]^r(x))dx = \underline{C}^r, \\ \int_a^b \overline{g}^r([y]^r(x))dx = \overline{C}^r, \\ \underline{y}^r(a) = \underline{y}^r_a, \quad \overline{y}^r(a) = \overline{y}^r_a, \\ \underline{y}^r(b) = \underline{y}^r_b, \quad \overline{y}^r(b) = \overline{y}^r_b, \\ \dot{\underline{y}}^r(a) = \dot{\underline{y}}^r_a, \quad \dot{\overline{y}}^r(a) = \dot{\overline{y}}^r_a, \\ \dot{\underline{y}}^r(b) = \dot{\underline{y}}^r_b, \quad \dot{\overline{y}}^r(b) = \dot{\overline{y}}^r_b, \end{aligned} \tag{9}$$

where $0 \le r \le 1$ and $0 \le w \le 1$.

The extremal of problem (9) can be obtained from Theorem 3.

Theorem 5 *The solution of the weighting problem* (9) *is Pareto optimal if the weighting coefficient is positive, that is,* $w > 0$. *Moreover, the unique solution of the weighting problem* (9) *is Pareto optimal.*

Proof. Let $(\underline{y}^{*r}, \overline{y}^{*r})$ be an optimal solution to problem (9) with $w > 0$. Suppose that $(\underline{y}^{*r}, \overline{y}^{*r})$ is not Pareto optimal. Then, there exists $(\underline{y}^r, \overline{y}^r)$ such that

$$\begin{cases} \underline{J}^r(\underline{y}^r, \overline{y}^r) \le \underline{J}^r(\underline{y}^{*r}, \overline{y}^{*r}) \\ \overline{J}^r(\underline{y}^r, \overline{y}^r) < \overline{J}^r(\underline{y}^{*r}, \overline{y}^{*r}) \end{cases} \quad \text{or} \quad \begin{cases} \underline{J}^r(\underline{y}^r, \overline{y}^r) < \underline{J}^r(\underline{y}^{*r}, \overline{y}^{*r}) \\ \overline{J}^r(\underline{y}^r, \overline{y}^r) \le \overline{J}^r(\underline{y}^{*r}, \overline{y}^{*r}) \end{cases}$$

$$\text{or} \quad \begin{cases} \underline{J}^r(\underline{y}^r, \overline{y}^r) < \underline{J}^r(\underline{y}^{*r}, \overline{y}^{*r}) \\ \overline{J}^r(\underline{y}^r, \overline{y}^r) < \overline{J}^r(\underline{y}^{*r}, \overline{y}^{*r}). \end{cases}$$

Since $w > 0$ we have

$$(w\underline{J}^r + (1-w)\overline{J}^r)(\underline{y}^r, \overline{y}^r) < (w\underline{J}^r + (1-w)\overline{J}^r)(\underline{y}^{*r}, \overline{y}^{*r}).$$

This contradicts the minimality of $(\underline{y}^{*r}, \overline{y}^{*r})$. Now, let $(\underline{y}^{*r}, \overline{y}^{*r})$ be the unique solution to (9). If $(\underline{y}^{*r}, \overline{y}^{*r})$ is not Pareto optimal, then

$$(w\underline{J}^r + (1-w)\overline{J}^r)(\underline{y}^r, \overline{y}^r) \le (w\underline{J}^r + (1-w)\overline{J}^r)(\underline{y}^{*r}, \overline{y}^{*r}).$$

This contradicts the uniqueness of $(\underline{y}^{*r}, \overline{y}^{*r})$. □

Therefore, by varying the weight over the $0 \leq w \leq 1$ ones obtains, in principle, different Pareto optimal solutions. The next theorem provides a necessary condition for Pareto optimality.

Theorem 6 *For a function* $(\underline{y}^{*r}, \overline{y}^{*r})$ *to be Pareto optimal to problem* (8)*, it is necessary to be a solution to the isoperimetric problems*

$$\int_a^b L^i([y]^r(x))dx \longrightarrow \min,$$
$$\int_a^b L^j([y]^r(x))dx = \int_a^b L^j([y^*]^r(x))dx,$$
$$\int_a^b \underline{g}^r([y]^r(x))dx = \underline{C}^r,$$
$$\int_a^b \overline{g}^r([y]^r(x))dx = \overline{C}^r,$$
$$\underline{y}^r(a) = \underline{y}^r_a, \quad \overline{y}^r(a) = \overline{y}^r_a,$$
$$\underline{y}^r(b) = \underline{y}^r_b, \quad \overline{y}^r(b) = \overline{y}^r_b,$$
$$\dot{\underline{y}}^r(a) = \dot{\underline{y}}^r_a, \quad \dot{\overline{y}}^r(a) = \dot{\overline{y}}^r_a,$$
$$\dot{\underline{y}}^r(b) = \dot{\underline{y}}^r_b, \quad \dot{\overline{y}}^r(b) = \dot{\overline{y}}^r_b,$$

for each $i = 1, 2$ *and* $i \neq j$*, where* $L^1 = \underline{L}^r$*,* $L^2 = \overline{L}^r$*, and* $r \in [0, 1]$.

Proof. Suppose that $(\underline{y}^{*r}, \overline{y}^{*r})$ is Pareto optimal. We define

$$\mathcal{C}_1 = \{(\underline{y}^r, \overline{y}^r) : \overline{J}^r(\underline{y}^r, \overline{y}^r) = \overline{J}^r(\underline{y}^{*r}, \overline{y}^{*r})\},$$
$$\mathcal{C}_2 = \{(\underline{y}^r, \overline{y}^r) : \underline{J}^r(\underline{y}^r, \overline{y}^r) = \underline{J}^r(\underline{y}^{*r}, \overline{y}^{*r})\}.$$

Then $(\underline{y}^{*r}, \overline{y}^{*r}) \in \mathcal{C}_k$, for $k = 1, 2$, so $\mathcal{C}_k \neq \phi$. If $(\underline{y}^{*r}, \overline{y}^{*r})$ does not minimize $\underline{J}^r(\underline{y}^r, \overline{y}^r)$ on the constrained set $\mathcal{C}_1$ and if $(\underline{y}^{*r}, \overline{y}^{*r})$ does not minimize $\overline{J}^r(\underline{y}^r, \overline{y}^r)$ on and $\mathcal{C}_2$, then there exists $(\underline{y}^r, \overline{y}^r)$ such that

$$\begin{cases} \underline{J}^r(\underline{y}^r, \overline{y}^r) < \overline{J}^r(\underline{y}^{*r}, \overline{y}^{*r}) \\ \overline{J}^r(\underline{y}^r, \overline{y}^r) = \overline{J}^r(\underline{y}^{*r}, \overline{y}^{*r}) \end{cases} \quad \text{and} \quad \begin{cases} \overline{J}^r(\underline{y}^r, \overline{y}^r) < \overline{J}^r(\underline{y}^{*r}, \overline{y}^{*r}) \\ \underline{J}^r(\underline{y}^r, \overline{y}^r) = \underline{J}^r(\underline{y}^{*r}, \overline{y}^{*r}). \end{cases}$$

This contradicts the Pareto optimality of $(\underline{y}^{*r}, \overline{y}^{*r})$. □

Example 1 (Example 7.1 of [11]) Find the optimal solution for the following fuzzy isoperimetric variational problem:

$$\begin{aligned} \tilde{J}(\tilde{y}(x)) &\approx \int_0^1 \dot{\tilde{y}}^2(x)dx \longrightarrow \min, \\ \tilde{I}(\tilde{y}(x)) &\approx \int_0^1 \tilde{y}(x)dx \approx < 0, 1, 3 >, \\ \tilde{x}(0) \approx 2 \approx < 2, 2, 2 >, &\quad \tilde{x}(1) \approx 4 \approx < 4, 4, 4 > . \end{aligned} \tag{10}$$

Solution. We assume that $\tilde{y}(x)$ is $[(1)-gH]-$differentiable function. In order to obtain the optimal solution of problem (10), From Theorem 4, we consider following biobjective variational problem

$$\begin{aligned}
&\left(\int_0^1 (\dot{\underline{y}}^r(x))^2dx, \int_0^1 (\dot{\overline{y}}^r(x))^2dx\right) \longrightarrow \min,\\
&\int_0^1 \underline{y}^r(x)dx = r,\\
&\int_0^1 \overline{y}^r(x)dx = 3-2r,\\
&\underline{y}(0)=2, \quad \overline{y}(0)=2,\\
&\underline{y}(1)=4, \quad \overline{y}(1)=4.
\end{aligned} \tag{11}$$

By Theorem 5, Pareto optimal solutions to problem (11) can be found by considering the family of problems

$$\begin{aligned}
&w\int_0^1 (\dot{\underline{y}}^r(x))^2dx + (1-w)\int_0^1 (\dot{\overline{y}}^r(x))^2dx \longrightarrow \min,\\
&\int_0^1 \underline{y}^r(x)dx = r,\\
&\int_0^1 \overline{y}^r(x)dx = 3-2r,\\
&\underline{y}(0)=2, \quad \overline{y}(0)=2,\\
&\underline{y}(1)=4, \quad \overline{y}(1)=4,
\end{aligned} \tag{12}$$

where $r\in[0,1]$ and $w\in[0,1]$. Let us now fix w. By Theorem 3, we have

$$F = w(\dot{\underline{y}}^r(x))^2 + (1-w)(\dot{\overline{y}}^r(x))^2 + \lambda_1\underline{y}^r(x) + \lambda_2\overline{y}^r(x)$$

and solution to problem (12) satisfies the Euler-Lagrange equations

$$\lambda_1 - 2\ddot{\underline{y}}^r.w = 0,$$

$$\lambda_2 - 2\ddot{\overline{y}}^r.(1-w) = 0.$$

Obviously, the latter differential equations are linear with constant coefficients, for fixed $w\in[0,1]$ and $r\in[0,1]$. We consider constants d_1 and d_2 such that $d_1=\frac{\lambda_1}{2w}$, $d_2=\frac{\lambda_2}{2(1-w)}$ and $w\neq 0,1$. Hence, by virtue of the classical differential equation theory, we may solve it analytically for fixed $w\in(0,1)$ and $r\in[0,1]$ to arrive at

$$\underline{y}^{*r}(x) = \frac{d_1}{2}x^2 + k_1x + k_2,$$

$$\overline{y}^{*r}(x) = \frac{d_2}{2}x^2 + p_1x + p_2.$$

Constants of integration $d_1; d_2; k_1; k_2; p_1; p_2$ might be given by the endpoint conditions as follows:

$$2 = \underline{y}^r(0) = k_2, \quad 2 = \overline{y}^r(0) = p_2,$$

$$4 = \underline{y}^r(1) = \frac{d_1}{2} + k_1 + k_2,$$

$$4 = \overline{y}^r(1) = \frac{d_2}{2} + p_1 + p_2.$$

Therefore we arrive at

$$\underline{y}^{*r}(x) = \frac{d_1}{2}x^2 + \left(\frac{4-d_1}{2}\right)x + 2,$$

$$\overline{y}^{*r}(x) = \frac{d_2}{2}x^2 + \left(\frac{4-d_2}{2}\right)x + 2.$$

On the other hand,

$$r = \int_0^1 \underline{y}^{*r}(x)dx = \int_0^1 \left(\frac{d_1}{2}x^2 + \left(\frac{4-d_1}{2}\right)x + 2\right)dx,$$

$$3 - 2r = \int_0^1 \overline{y}^{*r}(x)dx = \int_0^1 \left(\frac{d_2}{2}x^2 + \left(\frac{4-d_2}{2}\right)x + 2\right)dx,$$

which gives us

$$d_1 = 36 - 12r, \quad d_2 = 24r,$$

$$\underline{y}^{*r}(x) = (18-6r)x^2 + (6r-16)x + 2, \quad \overline{y}^{*r}(x) = 12rx^2 + (2-12r)x + 2.$$

One can easily show that $\underline{y}^{*r}(x)$ and $\overline{y}^{*r}(x)$ satisfy the Remark 1. Observe that $(\underline{y}^{*r}, \overline{y}^{*r})$ satisfies the necessary Pareto optimality conditions (See Theorem 6). Consider now the following isoperimetric problems:

$$\begin{aligned} &\int_0^1 (\dot{\underline{y}}^r(x))^2 dx \longrightarrow \min, \\ &\int_0^1 (\dot{\overline{y}}^r(x))^2 dx = \int_0^1 (\dot{\overline{y}}^{*r}(x))^2 dx, \\ &\int_0^1 \underline{y}^r(x)dx = r, \\ &\int_0^1 \overline{y}^r(x)dx = 3 - 2r, \\ &\underline{y}^r(0) = 2, \quad \overline{y}^r(0) = 2, \\ &\underline{y}^r(1) = 4, \quad \overline{y}^r(1) = 4. \end{aligned} \tag{13}$$

and

$$
\begin{aligned}
&\int_0^1 (\dot{\overline{y}}^r(x))^2 dx \longrightarrow \min,\\
&\int_0^1 (\dot{\underline{y}}^r(x))^2 dx = \int_0^1 (\dot{\underline{y}}^{*r}(x))^2 dx,\\
&\int_0^1 \underline{y}^r(x) dx = r,\\
&\int_0^1 \overline{y}^r(x) dx = 3 - 2r,\\
&\underline{y}^r(0) = 2, \quad \overline{y}^r(0) = 2,\\
&\underline{y}^r(1) = 4, \quad \overline{y}^r(1) = 4.
\end{aligned} \tag{14}
$$

where

$$\underline{y}^{*r}(x) = (18-6r)x^2 + (6r-16)x + 2, \quad \overline{y}^{*r}(x) = 12rx^2 + (2-12r)x + 2.$$

For the moment we consider only equation (13). The augmented function is

$$F = (\dot{\underline{y}}^r(x))^2 + \lambda_1(\dot{\overline{y}}^r(x))^2 + \lambda_2(\underline{y}^r(x)) + \lambda_3(\overline{y}^r(x)),$$

and the corresponding Euler–Lagrange equation gives

$$\lambda_2 - 2(\ddot{\underline{y}}^r(x))^2 = 0,$$

$$\lambda_3 - 2\lambda_1(\ddot{\overline{y}}^r(x))^2 = 0.$$

A solution to this equation is

$$\lambda_1 \neq 0, \ \lambda_2 = 72 - 24r, \ \lambda_3 = 48r\lambda_1,$$

$$\underline{y}^r(x) = \underline{y}^{*r}(x) = (18-6r)x^2 + (6r-16)x + 2$$

and

$$\overline{y}^r(x) = \overline{y}^{*r}(x) = 12rx^2 + (2-12r)x + 2.$$

Following the same arguments, one can show that

$$\underline{y}^r(x) = \underline{y}^{*r}(x) = (18-6r)x^2 + (6r-16)x + 2$$

and

$$\overline{y}^r(x) = \overline{y}^{*r}(x) = 12rx^2 + (2-12r)x + 2$$

is solution to (14). Therefore, by Theorem 6, $[\tilde{y}(x)]^r = [\tilde{y}^*(x)]^r$ is a candidate Pareto optimal solution to problem (11) and from Theorem 4, $\tilde{y}^*$ is a candidate optimal solution to problem (10) where $[\tilde{y}^*(x)]^r = [(18-6r)x^2 + (6r-16)x + 2, 12rx^2 + (2-12r)x + 2]$. This solution is shown in Figure 1, where the dashed lines are the $\overline{y}^r(x)$ and the doted lines are the $\underline{y}^r(x)$ for some $r \in [0, 1]$.

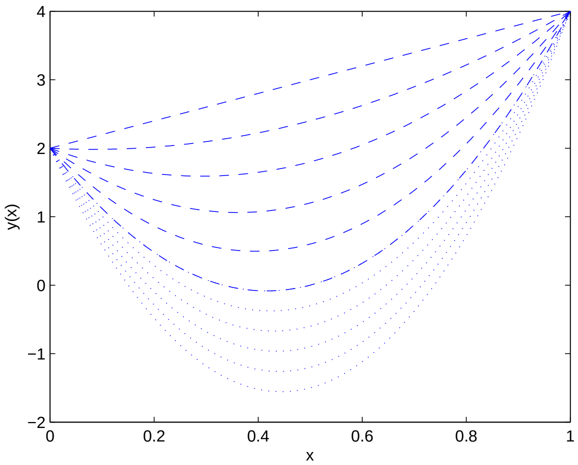

Fig. 1 The fuzzy optimal solution of Example 1.

4.1 Optimality for fuzzy variational problems

First, we review the theorem in classical calculus of variation. The well–known Euler–Lagrange equation for unconstraint variational problems is stated as follows.

Theorem 7 (See [22]) *Let* $J : \mathbf{C}^4[a,b] \longrightarrow \mathbb{R}$ *be a functional of the form,*

$$J(\mathbf{y}) = \int_a^b L(x, \mathbf{y}(x), \dot{\mathbf{y}}(x), \ddot{\mathbf{y}}(x))dx,$$

where L *has continuous partial derivatives of third order with respect to all its arguments. Let*

$$S = \{\mathbf{y} \in \mathbf{C}^4[a,b] : \mathbf{y}(a) = \mathbf{y}_a, \mathbf{y}(b) = \mathbf{y}_b, \dot{\mathbf{y}}(a) = \dot{\mathbf{y}}_a, \dot{\mathbf{y}}(b) = \dot{\mathbf{y}}_b\},$$

such that for $\mathbf{y} = (y_1, ..., y_n)$*, we have* $y_k \in C^4[a,b]$ *for* $k = 1, ..., n$*, and* $\mathbf{y}_a$*,* $\mathbf{y}_b$*,* $\dot{\mathbf{y}}_a$*,* $\dot{\mathbf{y}}_b$ *are given in* $\mathbb{R}^n$*. If* $\mathbf{y} \in S$ *is an extremal for* J*, then*

$$\frac{\partial L}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}_k}\right) + \frac{d^2}{dx^2}\left(\frac{\partial L}{\partial \ddot{y}_k}\right) = 0, \quad k = 1, ..., n, \tag{15}$$

for all $x \in [a,b]$.

Now we consider the following fuzzy variational problem:

$$\begin{aligned} \tilde{J}(\tilde{y}) \approx \int_a^b \tilde{L}(x, \tilde{y}(x), \dot{\tilde{y}}(x), \ddot{\tilde{y}}(x))dx &\longrightarrow \min, \\ \tilde{y}(a) \approx \tilde{y}_a, \quad \tilde{y}(b) &\approx \tilde{y}_b, \\ \dot{\tilde{y}}(a) \approx \dot{\tilde{y}}_a, \quad \dot{\tilde{y}}(b) &\approx \dot{\tilde{y}}_b, \end{aligned} \tag{16}$$

where $\underline{L}^r$ and $\overline{L}^r$ have continuous partial derivatives of third order with respect to all its arguments.

We say that $\tilde{y}^*$ is a solution of problem (16) if there exists no admissible curve $\tilde{y}(\neq \tilde{y}^*)$ such that $\tilde{J}(\tilde{y}) \prec \tilde{J}(\tilde{y}^*)$.

From the embedding Theorem 2, finally we consider the following variational problem

$$\begin{aligned}
&\Big(\underline{J}^r(\underline{y}^r, \overline{y}^r), \overline{J}^r(\underline{y}^r, \overline{y}^r)\Big) \to \min, \\
&\underline{y}^r(a) = \underline{y}^r_a, \quad \overline{y}^r(a) = \overline{y}^r_a, \\
&\underline{y}^r(b) = \underline{y}^r_b, \quad \overline{y}^r(b) = \overline{y}^r_b, \\
&\underline{\dot{y}}^r(a) = \underline{\dot{y}}^r_a, \quad \dot{\overline{y}}^r(a) = \dot{\overline{y}}^r_a, \\
&\underline{\dot{y}}^r(b) = \underline{\dot{y}}^r_b, \quad \dot{\overline{y}}^r(b) = \dot{\overline{y}}^r_b.
\end{aligned} \tag{17}$$

The fuzzy variational problem (16) is a special case of fuzzy isoperimetric problem (5). The proof of the following theorems are omitted as they are similar to those of Theorems 4 – 6.

Theorem 8 *If* $(\underline{y}^{*r}, \overline{y}^{*r})$ *is a Pareto optimal solution of the variational problem* (17) *then* $\tilde{y}^*$ *is an optimal solution of fuzzy variational problem* (16) *where* $[\tilde{y}^*]^r = (\underline{y}^{*r}, \overline{y}^{*r})$.

We obtain a sufficient condition for Pareto optimality by modifying the variational problem (17) into the following weighting problem:

$$\begin{aligned}
&w\int_a^b \underline{L}^r([y]^r(x))dx + (1-w)\int_a^b \overline{L}^r([y]^r(x))dx \to \min, \\
&\underline{y}^r(a) = \underline{y}^r_a, \quad \overline{y}^r(a) = \overline{y}^r_a, \\
&\underline{y}^r(b) = \underline{y}^r_b, \quad \overline{y}^r(b) = \overline{y}^r_b, \\
&\underline{\dot{y}}^r(a) = \underline{\dot{y}}^r_a, \quad \dot{\overline{y}}^r(a) = \dot{\overline{y}}^r_a, \\
&\underline{\dot{y}}^r(b) = \underline{\dot{y}}^r_b, \quad \dot{\overline{y}}^r(b) = \dot{\overline{y}}^r_b,
\end{aligned} \tag{18}$$

where $0 \le w \le 1$.

The extremal of problem (18) can be obtained from (15).

Theorem 9 *The solution of the weighting problem* (18) *is Pareto optimal if the weighting coefficient is positive, that is,* $w > 0$. *Moreover, the unique solution of the weighting problem* (18) *is Pareto optimal.*

Theorem 10 *For a function $(\underline{y}^{*r}, \overline{y}^{*r})$ to be Pareto optimal to problem (17), it is necessary to be a solution to the isoperimetric problems*

$$\int_a^b L^i([y]^r(x))dx \longrightarrow \min,$$

$$\int_a^b L^j([y]^r(x))dx = \int_a^b L^j([y^*]^r(x))dx, \quad j=1,2, \quad j \neq i,$$

$$\underline{y}^r(a) = \underline{y}^r_a, \quad \overline{y}^r(a) = \overline{y}^r_a,$$
$$\underline{y}^r(b) = \underline{y}^r_b, \quad \overline{y}^r(b) = \overline{y}^r_b,$$
$$\dot{\underline{y}}^r(a) = \dot{\underline{y}}^r_a, \quad \dot{\overline{y}}^r(a) = \dot{\overline{y}}^r_a,$$
$$\dot{\underline{y}}^r(b) = \dot{\underline{y}}^r_b, \quad \dot{\overline{y}}^r(b) = \dot{\overline{y}}^r_b,$$

where $L^1 = \underline{L}^r$, $L^2 = \overline{L}^r$, for each $i = 1, 2$ and $r \in [0,1]$.

Example 2 Find the optimal solution for the following fuzzy variational problem:

$$\begin{aligned} \tilde{J}(\tilde{y}(x)) \approx \int_0^1 -2.\tilde{y}(x) + (\ddot{\tilde{y}}(x))^2 dx \longrightarrow \min, \\ \tilde{y}(0) \approx \dot{\tilde{y}}(0) \approx < -1, 0, 1 >, \\ \tilde{y}(1) \approx \dot{\tilde{y}}(1) \approx < 0, 1, 2 > . \end{aligned} \tag{19}$$

Solution. First, we assume that $\tilde{y}(x)$ is $[(1,1)-gH]$-differentiable (*or* $[(2,2)-gH]$-differentiable) function. From Theorem 8, we cosider following biobjective variational problem

$$\begin{aligned} \left(\int_0^1 -2.\overline{y}^r(x) + (\ddot{\underline{y}}^r(x))^2 dx, \int_0^1 -2.\underline{y}^r(x) + (\ddot{\overline{y}}^r(x))^2 dx\right) \longrightarrow \min, \\ \underline{y}^r(0) = \dot{\underline{y}}^r(0) = r-1, \quad \overline{y}^r(0) = \dot{\overline{y}}^r(0) = 1-r, \\ \underline{y}^r(1) = \dot{\underline{y}}^r(1) = r, \quad \overline{y}^r(1) = \dot{\overline{y}}^r(1) = 2-r. \end{aligned} \tag{20}$$

We need to solve the following weighting variational problem

$$\begin{aligned} w\int_0^1 -2.\overline{y}^r(x) + (\ddot{\underline{y}}^r(x))^2 dx + (1-w)\int_0^1 -2.\underline{y}^r(x) + (\ddot{\overline{y}}^r(x))^2 dx \longrightarrow \min, \\ \underline{y}^r(0) = \dot{\underline{y}}^r(0) = r-1, \quad \overline{y}^r(0) = \dot{\overline{y}}^r(0) = 1-r, \\ \underline{y}^r(1) = \dot{\underline{y}}^r(1) = r, \quad \overline{y}^r(1) = \dot{\overline{y}}^r(1) = 2-r. \end{aligned} \tag{21}$$

From Theorem 7, Euler–Lagrange equation gives

$$\frac{d^2}{dx^2}(\ddot{\underline{y}}^r(x))) = \frac{1-w}{w}, \quad \frac{d^2}{dx^2}(\ddot{\overline{y}}^r(x))) = \frac{w}{1-w}.$$

Let us now fix $w = \frac{1}{2}$, from initial conditions, we arrive at

$$\underline{y}^{*r}(x) = \frac{1}{4!}x^4 + (2r - \frac{37}{12})x^3 + (-3r + \frac{121}{24})x^2 + (r-1)x + r - 1,$$

$$\overline{y}^{*r}(x) = \frac{1}{4!}x^4 + (-2r + \frac{11}{12})x^3 + (3r - \frac{23}{24})x^2 + (1-r)x + 1 - r.$$

One can check the conditions of Remark 1 are satisfied so $\tilde{y}^*(x)$ is fuzzy function. If we assume that $w = \frac{1}{3}$ then

$$\underline{y}^{*r}(x) = \frac{1}{12}x^4 + (2r - \frac{19}{6})x^3 + (-3r + \frac{61}{12})x^2 + (r-1)x + r - 1,$$

$$\overline{y}^{*r}(x) = \frac{1}{48}x^4 + (-2r + \frac{23}{24})x^3 + (3r - \frac{47}{48})x^2 + (1-r)x + 1 - r.$$

These solutions are shown in Figure 2 and Figure 3 for $w = \frac{1}{2}$ and $w = \frac{1}{3}$ respectively, where the dashed lines are the $\overline{y}^r(x)$ and the doted lines are the $\underline{y}^r(x)$ for some $r \in [0,1]$. Now we assume that $\tilde{y}(x)$ is $[(1,2)-gH]$-differentiable ($[(2,1)-gH]$-differentiable) function. We cosider following biobjective variational problem

$$\begin{gathered}\left(\int_0^1 -2.\overline{y}^r + (\ddot{\overline{y}}^r(x))^2 dx, \int_0^1 -2.\underline{y}^r + (\ddot{\underline{y}}^r(x))^2 dx\right) \longrightarrow \min,\\ \underline{y}^r(0) = \dot{\underline{y}}^r(0) = r - 1, \quad \overline{y}^r(0) = \dot{\overline{y}}^r(0) = 1 - r,\\ \underline{y}^r(1) = \dot{\underline{y}}^r(1) = r, \quad \overline{y}^r(1) = \dot{\overline{y}}^r(1) = 2 - r.\end{gathered} \tag{22}$$

We need to solve the following weighting variational problem

$$\begin{gathered} w\int_0^1 -2.\overline{y}^r + (\ddot{\overline{y}}^r(x))^2 dx + (1-w)\int_0^1 -2.\underline{y}^r + (\ddot{\underline{y}}^r(x))^2 dx \longrightarrow \min,\\ \underline{y}^r(0) = \dot{\underline{y}}^r(0) = r - 1, \quad \overline{y}^r(0) = \dot{\overline{y}}^r(0) = 1 - r,\\ \underline{y}^r(1) = \dot{\underline{y}}^r(1) = r, \quad \overline{y}^r(1) = \dot{\overline{y}}^r(1) = 2 - r.\end{gathered} \tag{23}$$

From Theorem 7, Euler–Lagrange equation gives

$$w(-2 + \frac{d^2}{dx^2}(2\ddot{\overline{y}}^r(x))) = 0, \quad (1-w)(-2 + \frac{d^2}{dx^2}(2\ddot{\underline{y}}^r(x))) = 0.$$

Let us now fix $w \neq 0, 1$, so we have

$$-2 + 2.(\overline{y}^r(x))^{(4)} = 0, \quad -2 + 2.(\underline{y}^r(x))^{(4)} = 0,$$

from initial conditions, we arrive at

$$\underline{y}^{*r}(x) = \frac{1}{4!}x^4 + (2r - \frac{37}{12})x^3 + (-3r + \frac{121}{24})x^2 + (r-1)x + r - 1,$$

$$\overline{y}^{*r}(x) = \frac{1}{4!}x^4 + (-2r + \frac{11}{12})x^3 + (3r - \frac{23}{24})x^2 + (1-r)x + 1 - r.$$

This solution is shown in Figure.2. Similar to Example 1, observe that $(\underline{y}^{*r}(x), \overline{y}^{*r}(x))$ satisfies the necessary Pareto optimality conditions (See Theorem 10).

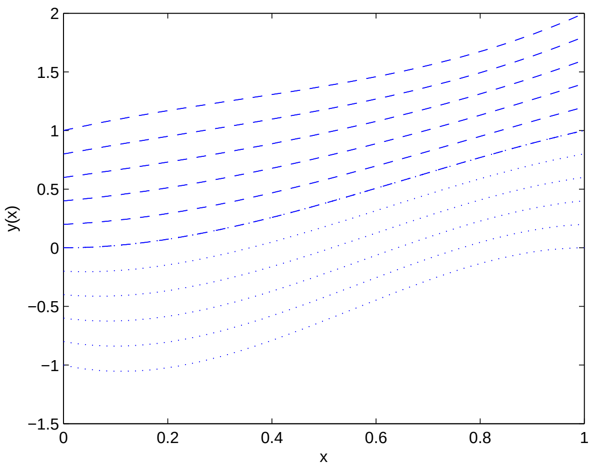

Fig. 2 The fuzzy optimal solution of Example 2 for $w = \frac{1}{2}$.

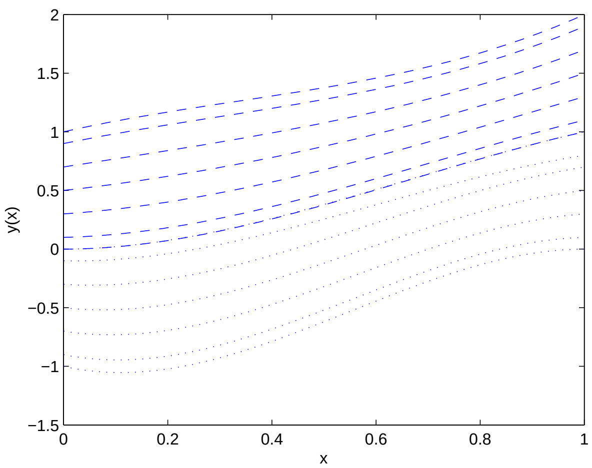

Fig. 3 The fuzzy optimal solution of Example 2 for $w = \frac{1}{3}$.

5 Conclusion

We proposed new concept of optimal solution for fuzzy variational problems. The main features of our optimality conditions were summarized and highlighted with two illustrative examples. As future work, we intend to derive optimal solutions for fuzzy optimal control problems.

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