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On Marginal Automorphisms of a Group Fixing the Certain Subgroup

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Abstract Let $\mathcal{W}$ be a variety of groups defined by a set W of laws and G be a finite p-group in $\mathcal{W}$. The automorphism α of a group G is said to be a marginal automorphism (with respect to W), if for all $x \in G$, $x^{-1}\alpha(x) \in W^*(G)$, where $W^*(G)$ is the marginal subgroup of G. Let M, N be two normal subgroups of G. By $Aut^M(G)$, we mean the subgroup of $Aut(G)$ consisting of all automorphisms which centralize G/M. $Aut_N(G)$ is used to show the subgroup of $Aut(G)$ consisting of all automorphisms which centralize N. We denote $Aut_N(G) \cap Aut^M(G)$ by $Aut_N^M(G)$. In this paper, we obtain a necessary and sufficient condition that $Aut_{w^*}(G) = Aut_{W^*(G)}^{W^*(G)}(G)$.

Keywords $\mathcal{W}$-nilpotent group · marginal automorphism · purely non-abelian group

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1 Introduction

2 Introduction and results

Let G be a group and F_∞ be the free group on the countable set $\{x_1, x_2, \cdots\}$ and W be a non-empty subset of F_∞. Suppose that

$$W(G) = \langle w(g_1, \cdots, g_r) | w = w(x_1, \cdots, x_r) \in W \text{ and } g_1, \cdots, g_r \in G\rangle$$

and

$$W^*(G) = \Big\{ g \in G \mid w(g_1, \cdots, g_{i-1}, g_i g, g_{i+1}, \cdots, g_r) = w(g_1, \cdots, g_r) \\ \text{for all } w \in W, \text{ for all } g_1, \cdots, g_r \in G \text{ and for all } 1 \le i \le r \Big\}.$$

For a group G, $W(G)$ and $W^*(G)$ are called, respectively, the verbal subgroup and the marginal subgroup of G with respect to W (see [8]). The verbal subgroup $W(G)$ is a fully invariant subgroup of G, and the marginal subgroup $W^*(G)$ is a characteristic subgroup of G.

We call an automorphism α of G a *marginal automorphism* with respect to W if $x^{-1}\alpha(x) \in W^*(G)$ for all $x \in G$.
The set of all marginal automorphisms of G forms a normal subgroup $Aut_{w^*}(G)$ of the automorphism group $Aut(G)$ of G. If we take $W = \{[x_1, x_2]\}$ where $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$, then $W(G) = G'$ and $W^*(G) = Z(G)$ where G' and $Z(G)$ are the commutator subgroup and the center of G, respectively.
In this case, $Aut_{w^*}(G)$ is denoted by $Aut_c(G)$ and the elements of $Aut_c(G)$ are called *central automorphism* of G. There are some well-known results about central automorphisms of G (for example see [1], [6] and [7]). Let M, N be two normal subgroups of G. By $Aut^M(G)$, we mean the subgroup of $Aut(G)$ consisting of all automorphisms which centralize G/M. $Aut_N(G)$ is used to show the subgroup of $Aut(G)$ consisting of all automorphisms which centralize N. We denote $Aut_N(G) \cap Aut^M(G)$ by $Aut_N^M(G)$.
Let be $\mathcal{W}$ a variety of groups defined by set W of laws then a group G is said to be $\mathcal{W}$-nilpotent if there exist a series

$$1 = G_0 \le G_1 \le \cdots \le C_k = G \tag{1}$$

such that $G_i \trianglelefteq G$ and $G_{i+1}/G_i \le W^*(G/G_i)$, for $0 \le i \le n$. The length of the shortest series (1) is the $\mathcal{W}$-nilpotent class of G.
For a finite p-group G define $\Omega_1(G) = \langle x \in G \mid x^p = 1\rangle$.
A non-abelian group G is called purely non-abelian if it has no non-trivial abelian direct factor. In this article such that group showed by PN-group. For any group H and abelian group K , $Hom(H, K)$ denote the group of all homomorphisms from H to K.
Azhdari and Malayeri [4] found necessary and sufficient condition that
$Aut_N^M(G) = Aut_c(G)$ and several corollaries where M , N are normal subgroups of a finite p-group G.
Through this article, $\mathcal{W}$ is a variety of groups and G is a finite p-group in

$\mathcal{W}$ which $W^*(G) \leq Z(G)$ and $G/W(G)$ is abelian also by the assumption $M \leq W^*(G)$ we have :

$$M = C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_s}} \tag{2}$$
$$W^*(G) = C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_{s'}}} \tag{3}$$
$$G/W(G) = C_{p^{d_1}} \times C_{p^{d_2}} \times \cdots \times C_{p^{d_r}} \tag{4}$$
$$\tag{5}$$

where $a_1 \geq a_2 \geq \cdots a_s > 0$, $b_1 \geq b_2 \geq \cdots b_{s'} > 0$ and also $d_1 \geq d_2 \geq \cdots d_r > 0$
Let t be the smallest integer between 1 and s such that $a_j = b_j$ for all $t \neq s$ and $t + 1 \leq j \leq s$. By this assumption our main results are the following.
Main theorem: Let $\mathcal{W}$ be a variety of group, G be finite p-group in $\mathcal{W}$ which is $\mathcal{W}$-nilpotent . Let M_1, M_2, N_1 and N_2 be normal subgroups of G such that $M_i \leq W^*(G) \cap N_i$ for $i = 1, 2$, $M_1 \leq M_2$ and $N_1 \leq N_2$. Then $Aut_{N_1}^{M_1}(G) = Aut_{N_2}^{M_2}(G)$ if and only if one of the following statements holds:

(i) $M_1 = M_2$ and $N_1 \leq W(G)G^{p^n}N_2$ where $exp(M_1) = p^n$ or
(ii) $N_1 = N_2$, $s = s'$ and $exp(G/W(G)N_1) \leq p^{a_t}$, where t is the smallest integer between 1 and s such that $a_j = b_j$.

By using the above notation we have the following corollary.
Corollary. Let G be a finite p-group which is $\mathcal{W}$-nilpotent group. Let M and N be two normal subgroups of G such that $M \leq W^*(G) \leq W(G)$. Then

(i) $Aut_N^M(G) = Aut_{W^*(G)}^{W^*(G)}(G)$ if and only if one of the following statements holds:
$M = W^*(G)$ and $N \leq W(G)G^{p^n}W^*(G)$ where $exp(W^*(G)) \leq p^{a_n}$ **or** $N = W^*(G)$, $s = s'$ and $exp(G/W(G)W^*(G)) \leq p^{a_t}$;
(ii) $Aut_N^M(G) = Aut_{w^*}(G)$ if and only if one of the following statements holds:
$M = W^*(G)$ and $N \leq W(G)G^{p^n}$ where $exp(W^*(G)) = p^{a_n}$ **or** $N \leq W(G)$, $s = s'$ and $exp(G/W(G)) \leq p^{a_t}$.

3 Preliminary results

Adeney and Yen [1, Theorem 1] prove that if G is a purely non-abelian, then there exist a bijection between $Aut_c(G)$ and $Hom(G/G', Z(G))$. Also Jamali and Mousavi in [6] prove that if G is a finite group such that $Z(G) \leq G'$ then $Aut_c(G) \cong Hom(G/G', Z(G))$.
Similarly Attar in [3] prove the following theorems about marginal automorphisms of a group G:

Theorem 1 *[3] Let G be a group and $\varnothing \neq W \subseteq F_\infty$ be the set of laws, Then $Aut_{w^*}(G)$ acts trivially on $W(G)$.*

Theorem 2 *[3] Let G be a purely non-abelian finite group and $\varnothing \neq W \subseteq F_\infty$ be the set of laws such that $W^*(G) \leq Z(G)$, then*

$$|Aut_{w^*}(G)| = |Hom\big(G/W(G), W^*(G)\big)|.$$

Theorem 3 *[3] Let G be a group and $\varnothing \neq W \subseteq F_\infty$ be the set of laws such that $W^*(G) \leq Z(G) \cap W(G)$, Then $Aut_{w^*}(G) \cong Hom\big(G/W(G), W^*(G)\big)$*

Proposition 1 *[3] Let G be a purely non-abelian finite group and $\varnothing \neq W \subseteq F_\infty$ be the set of laws such that $W^*(G)$ is abelian, Then*

(1) For each $\alpha \in Hom\big(G, W^*(G)\big)$ and $t \in W(G)$ we have $\alpha(t) = 1$;
(2) $Hom\big(G/W(G), W^*(G)\big) \cong Hom\big(G, W^*(G)\big)$.

By this proposition we have $Aut_{w^*}(G) \cong Aut_{W^*(G)}^{W(G)}(G)$ We recall that through this article $\mathcal{W}$ is a variety of groups and G is a finite p-group in $\mathcal{W}$ which $W^*(G) \leq Z(G)$ and $G/W(G)$ is abelian with abelian direct factor (4).

Theorem 4 *[3] Let $\mathcal{W}$ is a variety of groups and G is $\mathcal{W}$-nilpotent group and $1 \neq N \lhd G$, then $N \cap W^*(G) \neq 1$.*

Lemma 1 *[3] Let G be a finite PN-group and M , N be two normal subgroups of G such that $M \leq Z(G)$(in particular $M \leq W^*(G)$ since $W^*(G) \leq Z(G)$), then*

$$|Aut_N^M(G)| = |Hom\big(G/N, M\big)|.$$

Lemma 2 *[3] Let G be a group and M , N be two normal subgroups of G such that $M \leq Z(G) \cap N$, then*

$$Aut_N^M(G) \cong Hom\big(G/N, M\big)$$

and $Aut_N^M(G)$ is abelian.

Lemma 3 *[9] Let A and B be two finite abelian p-groups such that $A = C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_s}}$ where $a_1 \geq a_2 \geq \cdots a_s > 0$ and $B = C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_s}}$ where $b_1 \geq b_2 \geq \cdots b_s > 0$. Let $b_j \geq a_j$ for all j , $1 \leq j \leq s$ and $b_j > a_j$ for some such j. Let t be the smallest integer between 1 and s such that $a_j = b_j$ for all j such that $t+1 \leq j \leq s$. Then for any finite abelian p-group C, $|Hom(A,C)| < |Hom(B,C)|$ if and only if the exponent of C is at least p^{a_t+1}.*

4 Proof of the main results

Proof of The Main Theorem: We proceed with a series of steps. Let G be a finite p-group which is $\mathcal{W}$-nilpotent.

Step 1. Let $M \leq W^*(G)$ and $exp(M) = p^n$ and N be a normal subgroup of G. Then for all $f \in Hom(G/N, M)$ we have $kerf = W(G)G^{p^n}N/N$.

Proof. Clearly $W(G)G^{p^n}N/N \leq kerf$ for all $f \in Hom(G/N, M)$. To prove the converse inclusion, let $x \notin W(G)G^{p^n}N$. Since $M \leq W^*(G)$, then $Hom(G/N, M) \cong Hom(G/W(G)N, M)$. Put $\bar{G} = G/W(G)N$. $\bar{G}$ is a finite abelian p-group and so there exist $x_1, x_2, \cdots, x_t \in G$ such that $\bar{G} = \langle \bar{x}_1\rangle \times \langle \bar{x}_2\rangle \times \cdots \times \langle \bar{x}_t\rangle$ and $xW(G)N = x_1^{p^{s_1}} \cdots x_t^{p^{s_t}} W(G)N$ for suitable $s_i \geq 0$ (See [7 lemma 2.2]) Since $\bar{x} \notin W(G)G^{p^n}N/W(G)N$, $x_j^{p^{s_j}} \notin G^{p^n}$ for some $1 \leq j \leq t$ and therefor $s_j < n$. Now choose element $z \in M$ such that $|z| = min\{|\bar{x_j}|, p^n\}$, and define a homomorphism $f_z : \bar{x_j} \mapsto z$ from $\bar{G}$ to M. When $M \leq W^*(G)$ and $\bar{K}$ is a direct factor of $\bar{G}$ then any element f of $Hom(\bar{K}, M)$ induces an element $\bar{f}$ of $Hom(\bar{G}, M)$ which is trivial on the complement of $\bar{K}$ of $\bar{G}$. To simplify the notion , we will identify f with the corresponding homomorphism from $\bar{G}$ to M. We have $f_z(\bar{x}) = f_z(\bar{x_j}^{p^{s_j}}) = z_j^{p^s} \neq 1$. Thus $\bar{x} \notin kerf$ for $f \in Hom(G/N, M)$ and consequently the equality holds.

Step 2. Let N_1 , N_2 be two normal subgroups of G such that $N_2 \leq N_1$ and $M \leq W^*(G) \cap N_i$ for $i = 1, 2$. Then $Aut_{N_1}^M = Aut_{N_2}^M$ if and only if $N_1 \leq W(G)G^{p^n}N_2$ where $exp(M) = p^n$.

Proof. Since $N_2 \leq N_1$, $Aut_{N_1}^M \leq Aut_{N_2}^M$. Suppose that $N_1 \leq W(G)G^{p^n}N_2$, by using Step 1, $\bar{N_1} \leq kerf$ for all $f \in Hom(G/N_2, M)$. We have $Hom(G/N_2, M) \cong Hom(G/N_1N_2, M) \cong Hom(G/N_1, M)$, since $N_2 \leq N_1$. That is $|\ Aut_{N_1}^M(G)\ |=|\ Aut_{N_2}^M(G)\ |$ and hence $Aut_{N_1}^M(G) = Aut_{N_2}^M(G)$. Conversely, assume that $Aut_{N_1}^M(G) = Aut_{N_2}^M(G)$. Then $\alpha(n) = n$ for all $n \in N_1$ and $\alpha \in Aut_{N_2}^M(G)$. By Lemma 2 $\alpha^*(\bar{n}) = 1$ for all $\alpha^* \in Hom(G/N_2, M)$ and $n \in N_1$. Consequently by Step 1, $N_1 \leq W(G)G^{p^n}N_2$, as required.

Step 3. Let N be a normal subgroup of G and $M_1 < M_2 \leq W^*(G)G^{p^n}N$. Then $Aut_N^{M_1}(G) = Aut_N^{M_2}(G)$ if and only if $s = s'$ and $exp(G/W(G)N) \leq p^{a_t}$ where t is the smallest integer between 1 and s such that $a_j = b_j$ for all $t+1 \leq j \leq s$.

Proof. Since $M_1 \leq M_2$, $Aut_N^{M_1}(G) \leq Aut_N^{M_2}(G)$ by using Lemma 2 $Aut_N^{M_1}(G) = Aut_N^{M_2}(G)$ if and only if $Hom(G/N, M_1) \cong Hom(G/N, M_2)$. first assume that $Aut_N^{M_1}(G) = Aut_N^{M_2}(G)$, then clearly $s = s'$. So by applying Lemma 3 with $A = M_1$, $B = M_2$ and $C = W(G)N$ we get $exp(G/W(G)N) \leq p^{a_t}$, since if $exp(G/W(G)N) \geq p^{a_t+1}$, then we have $|\ Hom(G/N, M_1)\ |<|\ Hom(G/N, M_2)\ |$ which is a contradiction. Now suppose that $s = s'$ and $exp(G/W(G)N) \leq p^{a_t}$ then $|\ Hom(G/N, M_1)\ |=|\ Hom(G/N, M_2)\ |$ and therefor$Aut_N^{M_1}(G) \leq Aut_N^{M_2}(G)$.

Step 4. First assume that $Aut_{N_1}^{M_1}(G) = Aut_{N_2}^{M_2}(G)$. By Lemma 2 $Hom(G/N_1, M_1) \cong Hom(G/N_2, M_2)$. If $M_1 \leq M_2$ and $N_2 \leq N_1$ then by Lemma D [5] $Hom(G/N_1, M_1) \leq Hom(G/N_2, M_2)$. This contradiction implies $M_1 = M_2$ or $N_1 = N_2$. If $M_1 = M_2$ then by Step 2, $N_1 \leq W(G)G^{p^n}N_2$. Else, since $M_1 \neq M_2$, $N_1 = N_2$ then by Step 3, it follows that $s = s'$ and

$exp(G/W(G)N) \leq p^{a_t}$.
Conversely, if (i) or (ii) holds it is easy to see that $Hom(G/N_1, M_1) \cong Hom(G/N_2, M_2)$. On the other hand since $M_1 \leq M_2$ and $N_2 \leq N_1$, $Aut^{M_1}_{N_1}(G) \leq Aut^{M_2}_{N_2}(G)$ and consequently $Aut^{M_1}_{N_1}(G) = Aut^{M_2}_{N_2}(G)$, as required. □

Remark. Note that in the Proof of The Main Theorem, we use the conditions $M_i \leq W^*(G) \cap N_i$ only to prove the equality $\mid Aut^{M_i}_{N_i} \mid = \mid Hom(G/N_i, M) \mid$. So by Lemma 1 we may substitute this condition by "G be a PN-group". And by using the same argument, we can easily prove the following Theorem.

Theorem 5 *Let G be a finite p-group which is PN and $\mathcal{W}$-nilpotent group. Let M_1 and M_2 be two subgroups which is marginal subgroup and N_1 and N_2 be two normal subgroups of G such that $M_1 \leq M_2$ and $N_2 \leq N_1$. Then $Aut^{M_1}_{N_1}(G) \leq Aut^{M_2}_{N_2}(G)$ if and only if one of the following statements holds:*

(i) $M_1 = M_2$ and $N_1 \leq W(G)G^{p^n}N_2$ where $exp(M_1) \leq p^n$, or
(ii) $N_1 = N_2$, $s = s'$ and $exp(G/W(G)N_1) \leq p^{a_t}$ where t is the smallest integer between 1 and s such that $a_j = b_j$ for all $t + 1 \leq j \leq s$.

The Main Theorem has a number of important consequences. As a first application of this we get the following result.

Corollary 1 *Let G be a finite p-group which is $\mathcal{W}$-nilpotent group. Let M and N be two normal subgroups of G such that $M \leq W^*(G) \leq W(G)$. Then*

(i) $Aut^M_N(G) = Aut^{W^*(G)}_{W^*(G)}(G)$ if and only if one of the following statements holds:
$M = W^*(G)$ and $N \leq W(G)G^{p^n}W^*(G)$ where $exp(W^*(G)) \leq p^{a_n}$ ***or*** *$N = W^*(G)$, $s = s'$ and $exp(G/W(G)W^*(G)) \leq p^{a_t}$;*
(ii) $Aut^M_N(G) = Aut_{w^*}(G)$ if and only if one of the following statements holds:
$M = W^*(G)$ and $N \leq W(G)G^{p^n}$ where $exp(W^*(G)) = p^{a_n}$ ***or*** *$N \leq W(G)$, $s = s'$ and $exp(G/W(G)) \leq p^{a_t}$.*

Proof (i) The result follows immediately by applying The Main Theorem with $M_1 = M$, $N_1 = N$ and $M_2 = N_2 = W^*(G)$. To prove (ii), note that if M contained in the marginal subgroup of G, then $Aut^M_N(G) = Aut^M_{NW(G)}(G)$. Also $M \leq W^*(G) \leq N$ follows that $Aut_{w^*}(G) = Aut^{W^*(G)}_{W^*(G)}(G)$ and since G is a PN-group therefor, by applying Theorem 5 with $M_1 = M$, $N_1 = NW(G)$, $M_2 = W^*(G)$ and $N_2 = W(G)$, (ii) holds.
Conversely, if the firs part of (ii) holds then

$$\begin{aligned} Hom\big(G/N, M\big) \cong Hom\big(G/N, W^*(G)\big) &\cong Hom\big(G/W(G)N, W^*(G)\big) \\ &\cong Hom\big(G/W(G), W^*(G)\big) \end{aligned}$$

Now if the second part of (ii) holds then

$$Hom\big(G/N, M\big) \cong Hom\big(G/NW(G), W^*(G)\big) \quad \cong Hom\big(G/N, W^*(G)\big)$$

hence $\mid Hom\big(G/N, M\big) \mid = \mid Hom\big(G/W(G), W^*(G)\big) \mid$ or equivalently

$$Aut_N^M(G) = Aut_{w^*}(G)\ .$$

This corollary yields some following results.

Corollary 2 *Let G be a finite p-group which is $\mathcal{W}$-nilpotent group. Then $Aut_{w^*}(G) = Aut_{W^*(G)}^{W^*(G)}(G)$ if and only if $W^*(G) \leq W(G)G^{p^n}$ where $exp\big(W^*(G)\big) = p^n$.*

Proof It follows from Corollary 1 when $M = N = W^*(G)$.

Corollary 3 *Let G be a finite p-group which is $\mathcal{W}$-nilpotent group. Let M_1, M_2, N_1 and N_2 be normal subgroups of G such that $M_i \leq W^*(G) \cap N_i$ for $i = 1, 2$. Then $Aut_{N_1}^{M_1}(G) = Aut_{N_2}^{M_2}(G)$ if and only if one of the following statements holds:*

(i) $M_1 = M_2$ *and* $N_i \leq W(G)G^{p^{n_j}}N_j$ *for* $i = 1, 2$ *and* $i \neq j$*;*
(ii) $M_1 \leq M_2$,$s = s_1 = s_2$, $N_1 \leq N_2 \leq W(G)G^{p^{n_1}}N_1$ *and* $exp\big(G/W(G)N_2\big) \leq p^{a^{t_2}}$*;*
(iii) $M_2 \leq M_1$, $s = s_1 = s_2$, $N_2 \leq N_1 \leq W(G)G^{p^{n_2}}N_2$ *and* $exp\big(G/W(G)N_1\big) \leq p^{a^{t_1}}$*;*
(iv) $N_1 = N_2$, $s = s_1 = s_2$ *and* $exp\big(G/W(G)N_1\big) \leq p^{a^{t_i}}$ *for* $i = 1, 2$

Proof First assume that $Aut_{N_1}^{M_1}(G) = Aut_{N_2}^{M_2}(G)$. Therefore we have $Aut_{N_1}^{M_1}(G) = Aut_{N_1N_2}^{M_1\cap M_2}(G) = Aut_{N_2}^{M_2}(G)$. Clearly , $M_1 \cap M_2 \leq M_i$ and $N_i \leq N_1N_2$ for $i = 1, 2$ and so we may apply The Main Theorem. Since $Aut_{N_i}^{M_i}(G) = Aut_{N_1N_2}^{M_1\cap M_2}(G)$ for $i = 1, 2$ one of the following case happens.

(I) $M_i = M_1 \cap M_2$ and $N_1N_2 \leq W(G)G^{p^{n_i}}N_i$. So $M_i \leq M_j$ and $N_j \leq W(G)G^{p^{n_i}}N_i$ for $i \neq j$. Or,
(II) $N_i = N_1N_2$, $s = s_i$ and $exp\big(G/W(G)N_i\big) \leq p^{a^{t_i}}$. So $N_j \leq N_i$, $s = s_i$ and $exp\big(G/W(G)N_i\big) \leq p^{a^{t_i}}$ for $i \neq j$.

Therefore we have the following four cases:

(1) If for $i = 1, 2$ (I) holds, then $M_1 = M_2$ and $N_i \leq W(G)G^{p^{n_j}}N_j$ for $i, j = 1, 2$ and $i \neq j$ and hence (i) follows.
(2) If for $i = 1$ (I) and for $i = 2$ (II) happen, then $M_1 \leq M_2$, $N_2 \leq W(G)G^{p^{n_1}}N_1$ and so $s = s_2$, $N_1 \leq N_2$ and $exp\big(G/W(G)N_2\big) \leq p^{a^{t_2}}$. Since $a^{t_2} \leq n_1$, $G^{p^{n_1}} \leq W(G)N_2$ and $N_1 \leq N_2$ implies that $W(G)N_2 = W(G)G^{p^{n_1}}N_1$. Furthermore from $M_1 \leq M_2$, it follows that $s = s_1$ and consequently, $M_1 \leq M_2$, $s = s_1 = s_2$, $N_1 \leq N_2 \leq W(G)G^{p^{n_1}}N_1$ and $exp\big(G/W(G)N_2\big) \leq p^{a^{t_2}}$ and so in this case (ii) holds.

(3) If for $i = 1$ (II) and for $i = 2$ (I) happen, then with the argument similar to the case (2) we may conclude (ii) holds.
(4) Finally if $i = 1, 2$ (II) holds, then evidently $N_1 = N_2 = N_1N_2$, $s = s_1 = s_2$ and $exp\big(G/W(G)N_1\big) \leq p^{a^{t_i}}$, where $i = 1, 2$ that is (iv).

Conversely. First assume that (i) holds. Then we have $M_1 = M_2 = M$ and from $N_i \leq W(G)G^{p^{n_j}}N_j$. It follows that $Hom(G/N_j, M) \cong Hom(G/N_1N_2, M)$ for $i, j = 1, 2$ and $i \neq j$. Consequently $Aut^M_{N_i}(G) = Aut^M_{N_1N_2}(G)$ and hence $Aut^M_{N_1}(G) = Aut^M_{N_2}(G)$.
Now suppose that (ii) holds. Since $N_2 = N_1N_2$, $s = s_2$ and $exp\big(G/W(G)N_2\big) \leq p^{a^{t_2}}$, we have $Aut^{M_1\cap M_2}_{N_1N_2}(G) = Aut^{M_2}_{N_1N_2}(G) = Aut^{M_2}_{N_2}(G)$. Also $N_2 \leq W(G)G^{p^{n_1}}N_1$ concludes that $Aut^{M_1}_{N_1}(G) = Aut^{M_1}_{N_1N_2}(G) = Aut^{M_1\cap M_2}_{N_1N_2}(G)$ since $M_1 \leq M_2$. Therefore $Aut^{M_1}_{N_1}(G) = Aut^{M_2}_{N_2}(G)$. The case (ii) follows, by a similar argument.
Finally suppose that (iv) holds. So $N_1 = N_2 = N$, $s = s_1 = s_2$ and $exp\big(G/W(G)N_i\big) \leq p^{a^{t_i}}$ for $i = 1, 2$, it follows that $Aut^{M_i}_N(G) = Aut^{M_1\cap M_2}_N(G)$ and this completes the proof.

Note that here also, the condition " $M_i \leq W^*(G) \cap N_i$ for $i = 1, 2$" can be replaced by condition "G be a PN-group".

Corollary 4 *Let G be a finite p-group which is PN and $\mathcal{W}$-nilpotent group. Let M_1, M_2, N_1 and N_2 be two normal subgroups of G such that $M_i \leq W^*(G)$ for $i = 1, 2$. Then $Aut^{M_1}_{N_1}(G) = Aut^{M_2}_{N_2}(G)$ if and only if one of the following statements holds:*

(i) $M_1 = M_2$ *and* $N_i \leq W(G)G^{p^{n_j}}N_j$ *for* $i = 1, 2$ *and* $i \neq j$*;*
(ii) $M_1 \leq M_2$,$s = s_1 = s_2$, $N_1 \leq N_2 \leq W(G)G^{p^{n_1}}N_1$ *and* $exp\big(G/W(G)N_2\big) \leq p^{a^{t_2}}$*;*
(iii) $M_2 \leq M_1$, $s = s_1 = s_2$, $N_2 \leq N_1 \leq W(G)G^{p^{n_2}}N_2$ *and* $exp\big(G/W(G)N_1\big) \leq p^{a^{t_1}}$*;*
(iv) $N_1 = N_2$, $s = s_1 = s_2$ *and* $exp\big(G/W(G)N_1\big) \leq p^{a^{t_i}}$ *for* $i = 1, 2$

Another interesting equality is indicated by the following result.

Theorem 6 *Let G be a finite p-group which is PN and $\mathcal{W}$-nilpotent group. Let M, , N_1 and N_2 be normal subgroups of G such that $M \leq W^*(G)$. If the invariants of M (in the cyclic decomposition) are greather than or equal to $exp\big(G/W(G)N_i\big)$ for $i = 1, 2$ then $Aut^{M_1}_{N_1}(G) = Aut^{M_2}_{N_2}(G)$ if and only if $W(G)N_1 = W(G)N_2$.*

Proof Let $M = C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_s}}$ and $exp\big(G/W(G)N_i\big) = p^{n_i}$ for $i = 1, 2$. First assume that $N_2 \leq N_1$. By assumption $a_i \leq a_j$ for all $1 \leq j \leq s$ and $i = 1, 2$. Consequently we have

$$Hom(G/N_i, M) \cong Hom(G/N_i, C_{p^{a_1}}) \times \cdots \times Hom(G/N_i, C_{p^{a_s}}) \quad \cong (G/W(G)N_i)^n.$$

Therefore $Aut^M_{N_1}(G) = Aut^M_{N_2}(G)$ if and only if $G/W(G)N_1 = G/W(G)N_2$ or equivalently $W(G)N_1 = W(G)N_2$. Since $Aut^M_{N_1}(G) = Aut^M_{N_2}(G)$ if and only if $Aut^M_{N_i}(G) = Aut^M_{N_1N_2}(G)$ for $i = 1, 2$, the general case follows.

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