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Some Results on Baer's Theorem

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Abstract Baer has shown that, for a group G, finiteness of $G/Z_i(G)$ implies finiteness of $\gamma_{i+1}(G)$. In this paper we will show that the converse is true provided that $G/Z_i(G)$ is finitely generated. In particular, when G is a finite nilpotent group we show that $|G/Z_i(G)|$ divides $|\gamma_{i+1}(G)|^{d'_i(G)}$, where $d'_i(G) = (d(\frac{G}{Z_i(G)}))^i$.

Keywords Schur's Theorem · Lower central series · Upper central series

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1 Introduction

Let G be an arbitrary group. Let $Z_n(G)$ denotes the $(n+1)$-th term of the upper central series of G , and $\gamma_n(G)$ denotes the n-th term of the lower central series of G. A basic theorem of Schur (see [3, 10.1.4]) asserts that if the center of the group G has finite index, then the derived subgroup of G is finite. Moreover, some bounds for the order of the derived subgroup in terms of the index of the center were given by some authors, The best bound was given by Wiegold [4]. Thence finding some conditions under which the converse of Schur's theorem holds, have been interesting for some authors. There has been attempts to modify the statement and get conclusions. Some authors studied the situation under some extra conditions on the group. For example B. H. Neumann [1] proved that $G/Z(G)$ is finite if $\gamma_2(G)$ is finite and G is finitely generated. This result is recently generalized by P. Niroomand [2] by proving

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that $G/Z(G)$ is finite if $\gamma_2(G)$ is finite and $G/Z(G)$ is finitely generated. Now it raises various questions:

Question 1 Is there a generalization to higher terms of the upper and lower central series?

For generalizing to higher terms of the upper and lower central series, R. Baer (see for example [3, 14.5.1]) has proved that, if $G/Z_i(G)$ is finite, then $\gamma_{i+1}(G)$ is finite.

Question 2 Whether the converse of Baer's Theorem is true?

P. Hall (see for example [3, 14.5.3]) has proved a partial converse of Baer's theorem, that is, if $\gamma_{i+1}(G)$ is finite, then $G/Z_{2i}(G)$ is finite, thus proving that finite-by-nilpotent groups are nilpotent-by-finite. In this paper, as Theorems 3 and 4 show, we will give an affirmative answer to Question 2 under some conditions. In fact we will generalize the result of P. Niroomand and prove that finiteness of $G/Z_i(G)$ is equivalent to finiteness of $\gamma_{i+1}(G)$ provided that $G/Z_n(G)$ is finitely generated for some integers $n \leq i$.

2 Results

Let we start with the following interesting and useful result of Baer.

Theorem 1 (Baer, 14.5.1 of [3]) *If G is a group such that $G/Z_i(G)$ is finite. Then $\gamma_{i+1}(G)$ is finite.*

Theorem 2 *Let G be a finitely generated group. $\gamma_{i+1}(G)$ is finite if and only if $G/Z_i(G)$ is finite.*

Proof Let $a \in G$, since $\gamma_{i+1}(G) = [\gamma_i(G), G]$ is finite, the set of conjugates $\{a^b : b \in \gamma_i(G)\}$ is finite, so $C_{\gamma_i(G)}(a)$ has finite index in $\gamma_i(G)$. Since G is finitely generated, $\frac{\gamma_i(G)}{(\gamma_i(G)\bigcap Z(G))}$ is finite. Hence $\frac{\gamma_i(G)Z(G)}{Z(G)} = \gamma_i(G/Z(G))$ is finite. So by induction $\frac{G/Z(G)}{Z_{i-1}(G/Z(G))}$ is finite, and then $G/Z_i(G)$ is finite. Now Theorem 1 completes the proof.

The following result generalizes the main theorem of [2] to higher terms of the upper and lower central series, with the same conditions.

Theorem 3 *Let G be a group and let $G/Z(G)$ be finitely generated. Then $\gamma_{i+1}(G)$ is finite if and only if $G/Z_i(G)$ is finite.*

Proof By theorem 1, we should prove the sufficient. If $i = 1$, then the main theorem of [2] implies the result. Let $i > 1$ and $G/Z(G) = \langle x_1Z(G), \dots, x_nZ(G)\rangle$. As the proof of theorem 2, $C_{\gamma_i(G)}(a)$ has finite index in $\gamma_i(G)$ for any $a \in G$. Since $Z(G) = \bigcap_{i=1}^{n} C_G(x_i)$, $\frac{\gamma_i(G)}{(\gamma_i(G)\bigcap Z(G))}$ is finite. Hence $\frac{(\gamma_i(G)Z(G))}{Z(G)} = \gamma_i(G/Z(G))$ is finite. Now $\frac{G/Z(G)}{Z(G/Z(G))} = \frac{G}{Z_2(G)}$ is finitely generated so by induction $\frac{(G/Z(G))}{Z_{i-1}(G/Z(G))}$ is finite, and then $G/Z_i(G)$ is finite.

Lemma 1 *Let G be a group. Then $\gamma_n(G) \leq C_G(Z_n(G))$, where n is a positive integer.*

Proof It is an immediate consequence from 5.1.11 of [3].

Lemma 2 *Let G be a group and let $G/Z_n(G)$ be finitely generated, where n is a positive integer. Then $\gamma_{n+1}(G)$ is finite if and only if $G/Z_n(G)$ is finite.*

Proof If $n = 1$, then the result follows from Theorem 3. Now let $n \geq 2$ and $G/Z_n(G) = \langle \overline{x_1}, \overline{x_2}, \ldots, \overline{x_m} \rangle$. Then we have $G = \langle x_1, x_2, \ldots, x_m, Z_n(G) \rangle$ and $Z(G) = \bigcap_{i=1}^{m} C_G(x_i) \cap C_G(Z_n(G))$. Since $\gamma_{n+1}(G) = [\gamma_n(G), G]$ is finite, the set of conjugates $\{a^b : b \in \gamma_n(G)\}$ is finite, so $C_{\gamma_n(G)}(a)$ has finite index in $\gamma_n(G)$, for any $a \in G$. Now Lemma 1 implies that $\gamma_n(G) \cap Z(G) = \bigcap_{i=1}^{m} (C_G(x_i) \cap \gamma_n(G))$. Therefore $\gamma_n(G)/\gamma_n(G) \cap Z(G)$ is finite. It follows that $\gamma_n(G/Z(G))$ is finite. So by induction $G/Z_n(G)$ is finite. Now the proof is completed by theorem 1.

The following result is an interesting partially answer to the question whether the converse of Baer's Theorem is true.

Theorem 4 *Let G be a group and let $G/Z_n(G)$ be finitely generated, where n is a positive integer. Then $\gamma_{i+1}(G)$ is finite if and only if $G/Z_i(G)$ is finite, for all positive integers $i \geq n$.*

Proof We argue by induction on i. If $i = n$, then the result follows from Lemma 2. Now let $i > n$, then Lemma 1 implies that $\gamma_i(G) \leq C_G(Z_n(G))$. In the same way of the proof of Lemma 2, we obtain that $\gamma_i(G)/(\gamma_i(G) \cap Z(G)) \cong \gamma_i(G)Z(G)/Z(G)$ is finite. So $\gamma_i(G/Z(G))$ is finite. Since $\frac{G/Z(G}{Z_n(G/Z(G))} \cong G/Z_{n+1}(G) \cong \frac{G/Z_n(G)}{Z_{n+1}(G)/Z_n(G)}$ is finitely generated, by induction we have $\frac{G/Z(G}{Z_{i-1}(G/Z(G))}$ is finite. Consequently $G/Z_i(G)$ is finite. The converse follows from Theorem 1.

In the following we denote by $d(X)$, the minimal number of the generators of a group X. Let $d_i(G) = d(G/Z_i(G))$ for any group G, and $d'_i(G) = (d_i(G))^i$.

Lemma 3 *Let G be an arbitrary group and let n be a positive integer such that $d_n(G)$ and $\gamma_{n+1}(G)$ are finite. Then $|\gamma_n(G/Z(G))| \leq |\gamma_{n+1}(G)|^{d_n(G)}$.*

Proof If $n = 1$, then it follows from the main theorem of [2] that $|G/Z(G)| \leq |\gamma_2(G)|^{d_1(G)}$. Let $n \geq 2$, $m = d_n(G)$ and $G/Z_n(G) = \langle \overline{x_1}, \overline{x_2}, \ldots, \overline{x_m} \rangle$. Now as the proof of Lemma 2, $C_{\gamma_n(G)}(a)$ has finite index in $\gamma_n(G)$, for any $a \in G$. Hence Lemma 1 implies that:

$$\gamma_n(G) \cap Z(G) = \bigcap_{i=1}^{m} (C_G(x_i) \cap \gamma_n(G)) = \bigcap_{i=1}^{m} C_{\gamma_n(G)}(x_i).$$

It follows that:

$$|\gamma_n(G/Z(G))| = [\gamma_n(G) : \gamma_n(G) \cap Z(G)] = [\gamma_n(G) : \bigcap_{i=1}^{m} C_{\gamma_n(G)}(x_i)]$$

$$\leq \prod_{i=1}^{m}[\gamma_n(G) : C_{\gamma_n(G)}(x_i)] \leq |\gamma_{n+1}(G)|^m.$$

The following Theorem shows that the index $|G : Z_n(G)|$ can be bounded in terms of $|\gamma_{n+1}(G)|^{d'_n(G)}$.

Theorem 5 *Let G be an arbitrary group and let n be a positive integer such that $d_n(G)$ and $\gamma_{n+1}(G)$ are finite. Then $|G/Z_n(G)| \leq |\gamma_{n+1}(G)|^{d'_n(G)}$.*

Proof Let $m = d_n(G)$ and let $0 \leq j \leq n$ be an integer. It is easy to see that

$$m = d_n(G) = d_{n-1}(G/Z(G)) = \cdots = d_{n-j}(G/Z_j(G)).$$

Now by frequently using of Lemma 3 we have

$$|\gamma_n(G/Z(G))| \leq |\gamma_{n+1}(G)|^{d_n(G)};$$
$$|\gamma_{n-1}(G/Z_2(G))| \leq |\gamma_n(G/Z(G))|^{d_{n-1}(G/Z(G))};$$
$$\vdots$$
$$|\gamma_{n-j}(G/Z_{j+1}(G))| \leq |\gamma_{n+1-j}(G/Z_j(G))|^{d_{n-j}(G/Z_j(G))}.$$

Therefor we have

$$|G/Z_n(G)| = |\gamma_1(G/Z_n(G))| \leq |\gamma_2(G/Z_{n-1}(G))|^m \leq \cdots \leq |\gamma_{n+1}(G)|^{d'_n(G)}.$$

Now let p be a prime number and G a p-group. Since $|G/Z_n(G)|$ and $|\gamma_{n+1}(G)|^{d'_n(G)}$ are powers of p, Theorem 5 implies that $|G/Z_n(G)|$ divides $|\gamma_{n+1}(G)|^{d'_n(G)}$. therefor we have the following result.

Corollary 1 *Let n be a positive integer, p a prime and G an p- group such that $d_n(G)$ and $\gamma_{n+1}(G)$ are finite. Then $|G/Z_n(G)|$ divides $|\gamma_{n+1}(G)|^{d'_n(G)}$.*

Let G be a finite nilpotent group. Then $G = P_1 \times \cdots \times P_k$ where, $P_i, 1 \leq i \leq k$ is Sylow p_i-subgroup of G and p_i is a prime. It is known that $Z_i(G) = Z_i(P_1) \times Z_i(P_2) \times \cdots \times Z_i(P_k)$ and $\gamma_i(G) = \gamma_i(P_1) \times \gamma_i(P_2) \times \cdots \times \gamma_i(P_k)$. By using of Corollary 1 we conclude that $|G/Z_i(G)| = \prod_{i=1}^{k} |P_i/Z_i(P_i)|$ divides $\prod_{i=1}^{k} |\gamma_{i+1}(P_i)|^{d'_i(P_i)}$. Therefore we show that:

Corollary 2 *Let G be a finite nilpotent group and let n be a positive integer such that $d_n(G)$ and $\gamma_{n+1}(G)$ are finite. Then $|G/Z_n(G)|$ divides $|\gamma_{n+1}(G)|^{d'_n(G)}$.*

In the following result we generalize the Corollary 2 for other terms.

Corollary 3 *Let G be a nilpotent group and let n be a positive integer such that $d_n(G)$ and $\gamma_{n+1}(G)$ are finite, then $|G/Z_{n+j}(G)|$ divides $|\gamma_{j+1}(G)/(Z_n(G)\cap \gamma_{j+1}(G)|^{d'_j(G/Z_n(G))}$, where j is a positive integer.*

Proof By Theorem 5, $G/Z_n(G)$ is finite nilpotent group. Now we may apply corollary 2, for $G/Z_n(G)$ and j instead of G and n, respectively. Then the result follows.

The following example shows the finiteness conditions on the Theorem 5 is necessary.

Example 1 Let G be a group with generators $x_j, y_j, j > 1$ and z, subject to the relations $x_j^p = y_j^p = z^{p^i} = 1, [x_l, x_j] = [y_l, y_j] = 1$, for $k \neq j, [x_k, y_j] = 1$, and $[x_j, y_j, t_1, \ldots, t_r] = z^{p^r}$ where $t_s \in \{x_j, y_j\}$ for $s = 1, \ldots, r$ and $0 \leq r \leq i-1$. Then $Z_i(G) = \langle z \rangle$ and $\gamma_{i+1}(G) = \langle z^{p^{i-1}} \rangle$, but $G/Z_i(G)$ is infinite.

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