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Applications of the Natural-Adomian Decomposition Method to Estimate the Parameters of HIV Infection Model of $CD4^+$ T-Cells

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Abstract In this paper, we will obtain analytical approximate solutions of the HIV infection model of $CD4^+$ T-cells. This model corresponds to a class of nonlinear ordinary differential equation systems. To this end, we combine the Natural transform with the Adomian decomposition method for solving this model. The numerical results obtained by the suggested method are compared with the results obtained by other previous methods. These results indicate that this method agrees with other previous methods.

Keywords HIV infection of $CD4^+$ T-cells · Nonlinear system of differential equations · Natural-Adomian decomposition method · Numerical solution

Mathematics Subject Classification (2010) 34H05 · 60H15

1 Introduction

At the present time, almost 40 million people are infected by the human immunodeficiency virus (HIV) and about 16 million deaths have been caused by this disease, in the world. Scientists have made great efforts to treat this disease but there is still no cure for it. One of the consequences of infection by HIV is the selective depletion of $CD4^+$ T-cells which are commonly known as leukocytes or T-helper cells. Although HIV infects all cells, it has the most

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ruin on the $CD4^+$ T-cells and uses these cells in order to propagate. With the help of these cells, which are abundant in white blood cells of the immune system, the body fights against diseases. In people infected by HIV, the number of $CD4^+$ T-cells decreases, and therefore, the resistance of the immune system decreases. So, the body cannot fight against other infectious diseases such as cancers.

Because of the great importance of this disease, we solve the model of this disease in this paper. Several nonlinear mathematical models have been proposed to describe infection by HIV. In 1989, a model for the infection of the human immune system by HIV was developed by Perelson [7]. Perelson characterized this model by a system of nonlinear differential equations as

$$\begin{cases} \frac{dT}{dt} = q - \alpha T + rT\left(1 - \frac{T+I}{T_{max}}\right) - kVT, \\ \\ \frac{dI}{dt} = kVT - \beta I, \\ \\ \frac{dV}{dt} = N^*\beta I - \gamma V, \end{cases} \tag{1}$$

with the initial conditions

$$T(0) = r_1, \qquad I(0) = r_2, \qquad V(0) = r_3. \tag{2}$$

Table 1 summarizes the meanings of variables and parameters. The stability and existence of this model were discussed in [12].

Table 1: List of variables and parameters [11, 7, 5].

Variables and Parameters	Meaning
T	The concentration of susceptible $CD4^+$ T-cells
I	$CD4^+$ T-cells infected by the HIV viruses
V	Free HIV virus particles in the blood
q	The rate of production of $CD4^+$ T-cells and is constant
α	The natural turnover rate of T-cells
r	The Growth rate of $CD4^+$ T-cells concentration
T_{max}	The maximum level of $CD4^+$ T-cells in the human body
k	The rate of infection of T-cells by virus
β	The per capita rate of disappearance of infected cells
N^*	The average number of virus particles produced by an infected T-cell
γ	The death rate of virus particles

For solving numerically model (1), Merdan et al. have used the Padé approximate and the variational iteration method (VIM) [4], Venkatesh et al. in [10], have applied the Legendre wavelet method (LWM), and Ongun has used the Laplace Adomian decomposition method (LADM) [6]. In this paper, the Natural-Adomian decomposition method (N-ADM) is applied to solve a model for HIV infection of $CD4^+$ T-cells (1) with the initial conditions (2).

The structure of this paper is as follows: In Section 2, we describe the preliminaries related to Natural transform (N-transform). In Section 3, the application and procedure of the N-ADM, are presented. The numerical results are reported in Section 4. Also, concluding remarks are made in Section 5.

2 Preliminaries related to N-transform

the two sided N-transform of the function $\psi(t) \in (-\infty, \infty)$ may be defined on the both sides of real line $t \in (-\infty, +\infty)$,

$$\mathrm{N}[\psi(t)] = \int_{-\infty}^{\infty} e^{-st}\psi(ut)\,dt, \qquad s, u \in (-\infty, \infty). \tag{3}$$

The equation (3), is defined precisely

$$\begin{aligned}\mathrm{N}[\psi(t)] &= \int_{-\infty}^{0} e^{-st}\psi(ut)\,dt + \int_{0}^{\infty} e^{-st}\psi(ut)\,dt\\ &= \mathrm{N}^{-}[\psi(t)] + \mathrm{N}^{+}[\psi(t)]\\ &= \mathrm{N}[\psi(t)H(-t)] + \mathrm{N}[\psi(t)H(t)].\end{aligned}$$

Now, for the function $\psi(t)H(t)$ where $H(t)$ is unit step function, the N-transform of $\psi(t) \in (0, \infty)$ is defined by [3],

$$\mathrm{N}^{+}[\psi(t)] = \mathcal{R}(s,u) = \int_{0}^{\infty} e^{-st}\psi(ut)\,dt, \qquad s, u \in (0, \infty), \tag{4}$$

provided the function $\psi(t)H(t)$ is deflned in the set

$$\mathscr{H} = \Big\{\psi(t) \mid \exists\, \mathcal{M}, \kappa_1, \kappa_2 > 0,\ |\psi(t)| < \mathcal{M}e^{\frac{|t|}{\kappa_i}},\ \text{if } t \in (-1)^i \times [0,\infty),\ i = 1,2\Big\}.$$

The integral equation (4), converges to Laplace transform [9] when $u \equiv 1$ [3], and into Sumudu transform [1] for $s \equiv 1$ [3]. Meanwhile, we recall the definition and theorems of N-transform without proof as mentioned in [3,8].

Definition 1 The N-transform is a linear operator. That is if $\mathcal{R}_1(s,u)$ and $\mathcal{R}_2(s,u)$ be the N-transforms of $\psi(t)$ and $\varphi(t)$, respectively, and c_1 and c_2 are non-zero constants, then

$$\mathrm{N}^{+}[c_1\psi(t) \pm c_2\varphi(t)] = c_1\mathrm{N}^{+}[\psi(t)] \pm c_2\mathrm{N}^{+}[\varphi(t)] = c_1\mathcal{R}_1(s,u) \pm c_2\mathcal{R}_2(s,u).$$

Theorem 1 *If $\mathcal{R}(s,u)$ is the N-transform and $F(s)$ is the Laplace transform of the function $\psi(t) \in \mathscr{H}$, then*

$$N^{+}[\psi(t)] = \mathcal{R}(s,u) = \frac{1}{u}\int_{0}^{\infty} e^{-\frac{st}{u}}\psi(t)\,dt = \frac{1}{u}F\Big(\frac{s}{u}\Big).$$

Theorem 2 *If $N^{+}[\psi(t)] = \mathcal{R}(s,u)$, then $N^{+}[\psi'(t)] = \frac{s}{u}\mathcal{R}(s,u) - \frac{\psi(0)}{u}$.*

3 Application and procedure of the N-ADM

In this section, to illustrate the basic idea of this method, we consider the system of the nonlinear differential equations (1) with the initial conditions (2).

Applying the N-transform (denoted in this paper as N^+) to both sides of the equations of system (1) and by using the property of the N-transform, we get

$$\begin{cases} \mathrm{N}^+(T) = \frac{T(0)}{s} + q\frac{u}{s^2} + \frac{u}{s}\mathrm{N}^+\Big[(r-\alpha)T - \frac{r}{T_{max}}T^2 - \frac{r}{T_{max}}TI - kVT\Big], \\ \mathrm{N}^+(I) = \frac{I(0)}{s} + \frac{u}{s}\mathrm{N}^+\Big[kVT - \beta I\Big], \\ \mathrm{N}^+(V) = \frac{V(0)}{s} + \frac{u}{s}\mathrm{N}^+\Big[N^*\beta I - \gamma V\Big]. \end{cases} \tag{5}$$

Applying the inverse N-transform to both sides of equations (5), we have

$$\begin{cases} T(t) = r_1 + qt + \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^+\Big[(r-\alpha)T - \frac{r}{T_{max}}T^2 - \frac{r}{T_{max}}TI - kVT\Big]\right], \\ I(t) = r_2 + \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^+\Big[kVT - \beta I\Big]\right], \\ V(t) = r_3 + \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^+\Big[N^*\beta I - \gamma V\Big]\right]. \end{cases} \tag{6}$$

Now, we assume an infinite series solutions for the unknown functions $T(t)$, $I(t)$, and $V(t)$ are given in the form

$$\begin{cases} T(t) \simeq \tilde{T}(t) = \sum_{n=0}^{\infty} T_n(t), \\ I(t) \simeq \tilde{I}(t) = \sum_{n=0}^{\infty} I_n(t), \\ V(t) \simeq \tilde{V}(t) = \sum_{n=0}^{\infty} V_n(t), \end{cases} \tag{7}$$

and the nonlinear term can be decomposed as

$$TI = \sum_{n=0}^{\infty} A_n(t), \quad VT = \sum_{n=0}^{\infty} B_n(t), \quad T^2 = \sum_{n=0}^{\infty} C_n(t), \tag{8}$$

where A_n, B_n, and C_n are the Adomian polynomials which can be computed by the following formulas ([2])

$$\begin{aligned} A_n(t) &= \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\Big(\sum_{j=0}^{n}\lambda^j T_j\Big)\Big(\sum_{j=0}^{n}\lambda^j I_j\Big)\right]_{\lambda=0}, \\ B_n(t) &= \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\Big(\sum_{j=0}^{n}\lambda^j V_j\Big)\Big(\sum_{j=0}^{n}\lambda^j T_j\Big)\right]_{\lambda=0}, \\ C_n(t) &= \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\Big(\sum_{j=0}^{n}\lambda^j T_j\Big)^2\right]_{\lambda=0}. \end{aligned} \tag{9}$$

Substituting (7) and (8) in (6), we have

$$\begin{aligned} &\begin{cases} T_0(t) = r_1 + qt, \\ I_0(t) = r_2 \\ V_0(t) = r_3 \end{cases} \\ &\begin{cases} T_1(t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\Big[(r-\alpha)T_0(t) - \frac{r}{T_{max}}C_0(t) - \frac{r}{T_{max}}A_0(t) - kB_0(t)\Big]\right], \\ I_1(t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\Big[kB_0(t) - \beta I_0(t)\Big]\right], \\ V_1(t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\Big[N^{*}\beta I_0(t) - \gamma V_0(t)\Big]\right], \end{cases} \\ &\vdots \\ &\begin{cases} T_{n+1}(t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\Big[(r-\alpha)T_n(t) - \frac{r}{T_{max}}C_n(t) - \frac{r}{T_{max}}A_n(t) - kB_n(t)\Big]\right], \\ I_{n+1}(t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\Big[kB_n(t) - \beta I_n(t)\Big]\right], \\ V_{n+1}(t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\Big[N^{*}\beta I_n(t) - \gamma V_n(t)\Big]\right], \end{cases} \end{aligned} \tag{10}$$

for $n \geq 0$. According to the equations (10), the terms of $T_1, T_2, \dots, I_1, I_2, \dots,$ and $V_1, V_2, \dots,$ can be calculated recursively. As a result, we can write the solution

$$\tilde{T}(t) = T_0(t) + T_1(t) + T_2(t) + \cdots ,$$
$$\tilde{I}(t) = I_0(t) + I_1(t) + I_2(t) + \cdots ,$$
$$\tilde{V}(t) = V_0(t) + V_1(t) + V_2(t) + \cdots .$$

4 Numerical results

In this section, we apply the present method to get the numerical solution of model (1) on the interval $[0, 1]$, with the following initial conditions and values:

$$r_1 = r_3 = 0.1, \quad r_2 = 0, \quad q = 0.1, \quad \alpha = 0.02, \quad r = 3,$$
$$T_{max} = 1500, \quad k = 0.00027, \quad \beta = 0.3, \quad N^* = 10, \quad \gamma = 2.4.$$

Remark 1 To check the accuracy of the method, we can do as follows:
Since the solutions (7) are approximate solutions of the system (1), when the functions $\tilde{T}(t)$, $\tilde{I}(t)$, $\tilde{V}(t)$, and their derivatives are substituted in this system, the resulting equation must be satisfied approximately; that is for $t_s \in [0, 1]$, $s = 1, 2, \ldots$

$$\begin{cases} \mathrm{E}_T(t_s) = \left|\tilde{T}'(t_s) - q + \alpha\tilde{T}(t_s) - r\tilde{T}(t_s)\left(1 - \frac{\tilde{T}(t_s)+\tilde{I}(t_s)}{T_{max}}\right) + k\tilde{V}(t_s)\tilde{T}(t_s)\right|, \\ \mathrm{E}_I(t_s) = \left|\tilde{I}'(t_s) - k\tilde{V}(t_s)\tilde{T}(t_s) + \beta\tilde{I}(t_s)\right|, \\ \mathrm{E}_V(t_s) = \left|\tilde{V}'(t_s) - N^*\beta\tilde{I}(t_s) + \gamma\tilde{V}(t_s)\right|. \end{cases}$$

We compare the present method with other previous methods in Tables 2–4. Also, the graphs of approximate solutions and error calculated for six iterates, are shown in Figs. 1 and 2.

Table 2: Numerical comparison for $T(t)$.

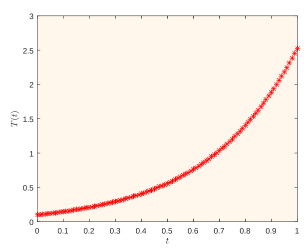
t	VIM [4]	LWM [10]	LADM [6]	Runge-Kutta	N-ADM
0	0.1	0.1	0.1	0.1	0.1
0.2	0.20880732	0.20880732	0.20880727	0.20880808	0.20880750
0.4	0.40613466	0.40612456	0.40610526	0.40624054	0.40615815
0.6	0.76245304	0.76414764	0.76114677	0.76442389	0.76285744
0.8	1.39788059	1.39777462	1.37731986	1.41404683	1.40093769
1	2.50674667	2.55714623	2.32916976	2.59159480	2.52148158

Table 3: Numerical comparison for $I(t)$.

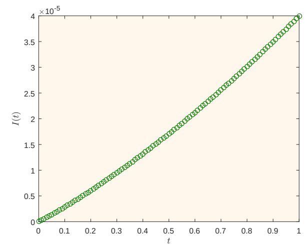
t	VIM [4]	LWM [10]	LADM [6]	Runge-Kutta	N-ADM
0	0	0	0	0	0
0.2	$0.60326347e-05$	$0.60327047e-05$	$0.60327073e-05$	$0.60327022e-05$	$0.60327019e-05$
0.4	$0.13148785e-04$	$0.13167845e-04$	$0.13159162e-04$	$0.13158341e-04$	$0.13158267e-04$
0.6	$0.21014172e-04$	$0.21126288e-04$	$0.21268369e-04$	$0.21223785e-04$	$0.21221993e-04$
0.8	$0.27951304e-04$	$0.29981397e-04$	$0.30069187e-04$	$0.30177420e-04$	$0.30160383e-04$
1	$0.24315623e-04$	$0.32876543e-04$	$0.39873654e-04$	$0.40037815e-04$	$0.39940702e-04$

Table 4: Numerical comparison for $V(t)$.

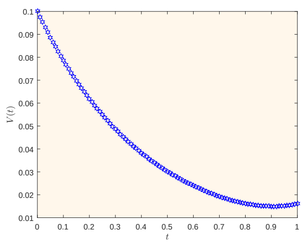
t	VIM [4]	LWM [10]	LADM [6]	Runge-Kutta	N-ADM
0	0.1	0.1	0.1	0.1	0.1
0.2	0.06187995	0.06187991	0.06187996	0.06187984	0.06187995
0.4	0.03830820	0.03832342	0.03831325	0.03829489	0.03830818
0.6	0.02392029	0.02381099	0.02439174	0.02370455	0.02391989
0.8	0.01621704	0.01621390	0.00996722	0.01468036	0.01621286
1	0.01608419	0.01605042	0.00330508	0.00910084	0.01605750



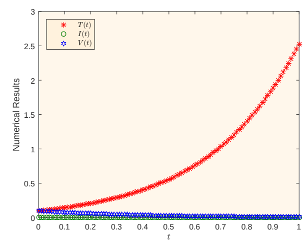
(a) N-ADM method for determination of $T(t)$



(b) N-ADM method for determination of $I(t)$

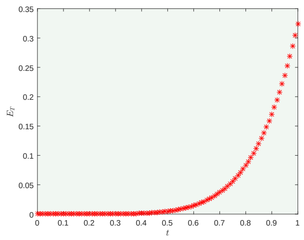


(c) N-ADM method for determination of $V(t)$

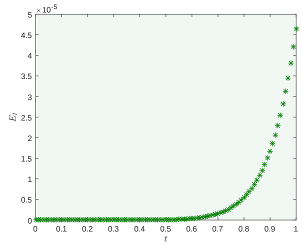


(d) N-ADM method for determination of $T(t)$, $I(t)$, and $V(t)$

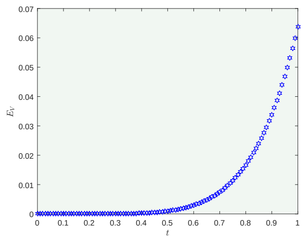
Fig. 1: Numerical results for determination of $T(t)$, $I(t)$, and $V(t)$ for six iterates.



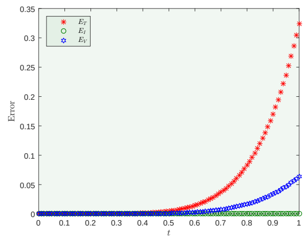
(a) Error calculated for $T(t)$



(b) Error calculated for $I(t)$



(c) Error calculated for $V(t)$



(d) Error calculated for $T(t)$, $I(t)$, and $V(t)$

Fig. 2: Error calculated for $T(t)$, $I(t)$, and $V(t)$ for six iterates.

5 Conclusion

In this paper, we used the N-ADM to solve a model for HIV infection of $CD4^+$ T-cells which is a class of nonlinear ordinary differential equation systems. We have demonstrated the accuracy and efficiency of the present method with an example. The comparisons between the present method and some existing methods such as VIM, LWM, LADM, and the Runge-Kutta, in Tables 2–4, show that the present method is in agreement with other methods.

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