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Some Remarks on c-Isoclinic Pairs of Filippov Algebras

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Abstract In this paper, we study the notion of c-isoclinism for the pairs of Filippov algebras. Also, we give an equivalent condition for pairs of Filippov algebras to be c-isoclinic. In particular, it is shown that two Filippov algebras are c-isoclinic if and only if then each of them can be constructed from another by using the operations of forming direct sums, taking subalgebras, and factoring Filippov algebras. Moreover, we introduce the concept of c-perfect pair of Filippov algebras and obtain some relations between c-isoclinic and c-perfect pairs of Filippov algebras.

Keywords Filippov algebras · c-isoclinism · c-perfect

Mathematics Subject Classification (2010) 17B05 · 17B99

1 Introduction

In 1940, Hall [5] introduced an equivalence relation on the class of all groups, called isoclinism, which is weaker than isomorphism. It was generalized to n-isoclinism and isologism with respect to a given variety of groups by several authors (See [2, 8, 9, 11, 12] for more information).

In [22], the concept of isoclinism has been extended to pairs of groups. Later, many authors studied the concepts of n-isoclinism and isologism for pairs of groups in [7, 6, 10].

In 1994, Moneyhun [15] gave a Lie algebra analogue of isoclinism. The concepts of isoclinism and n-isoclinism for Lie algebras was studied in [19–21]. In 2009, Moghaddam and parvaneh [13] extended the notion of isoclinism to

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the pairs of Lie algebras and gave some structural properties of the notion. This concept was extended by Moghaddam et al. [14] in 2014. They studied the notion of relative n-isoclinism between two pairs of Lie algebras and gave some properties of this notion. Also, the second author and Safa [1] investigated this notion for the pairs $(\mathrm{Ker}\sigma_1, M_1)$ and $(\mathrm{Ker}\sigma_2, M_2)$, where

$$\sigma_i : M_i \to L, (i = 1, 2),$$

are two c-covers of a pair (N, L) of Lie algebras.

Saeedi and Veisi [18] generalized the notion of isoclinism to n-Lie algebras and Eshrati et al. [3] proved that the notion of isoclinism and isomorphism are equivalent for any two n-Lie algebras of the same dimensions. Also, the first author and saeedi [17] studied special drivations of isoclinic n-Lie algebras. The concept of isoclinism is generalized by Mousavi and Moghaddam [16] for the pairs of n-Lie algebras. They proved that each equivalence class of isoclinic pairs of n-Lie algebras contains a stem pair of n-Lie algebras, which has minimal dimension amongst the finite dimensional pairs of n-Lie algebras.

In this paper, we introduce the concept of c-isoclinism for the class of all pairs of n-Lie (Filippov) algebras (M, L), where M is an ideal of L. The structure of the paper is as follows. In section 2, we present fundumental notions which are required in this paper. In section 3, the concept of c-isoclinism for pairs of Filippov algebras are studied and we give an equivalent condition for pairs of Filippov algebras to be c-isoclinic. It is shown that if two Filippov algebras L_1 and L_2 are c-isoclinic then L_1 can be constructed from L_2 using the operations of forming direct sums, taking subalgebras, and factoring Filippov algebras. In section 4, we introduce the concept of c-perfect pairs of Filippov algebars. Also, we obtain some relations between c-isoclinic and c-perfect pairs of Filippov algebras.

2 Fundamental notions

In 1985, Filippov [4] introduced the concept of Filippov algebras and classified the $(n + 1)$-dimensional Filippov algebras over an algebraically closed field of characteristic zero.

All Filippov algebras are considered over a fixed field F and $[-, \ldots, -]$ denotes the n-Lie bracket. A Filippov algebra is a vector space L over field F on which an n-ary multilinear and skew-symmetric operation $[x_1, \ldots, x_n]$ is defined satisfying the *generalized Jacobi identity*

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n].$$

Clearly, such an algebra becomes an ordinary Lie algebra when $n = 2$.

A subspace A of a Filippov algebra L satisfying $[x_1, \ldots, x_n] \in A$ for any $x_1, \ldots, x_n \in A$ is called a *subalgebra* of L. Let $A_1, \ldots, A_n$ be subalgebras of a

Filippov algebra L. Denote by $[A_1, \ldots, A_n]$ the subspace of L generated by all $[x_1, \ldots, x_n]$, where $x_i \in A_i$ for $i = 1, 2, \ldots, n$. The subalgebra

$$L^2 = [\underbrace{L, L, \ldots, L}_{n-times}],$$

is called the *derived algebra* of L. An *ideal* I of a Filippov algebra L is a subspace of L such that $[I, \underbrace{L, \ldots, L)}_{(n-1)-times}] \subseteq I$. The ideal

$$Z(L) = \{x \in L \mid [x, y_1, \ldots, y_{n-1}] = 0, \forall y_1, \ldots, y_{n-1} \in L\},$$

is called the *center* of L. Also, L^i is defined inductively by [18],

$$L^1 = L, \qquad \text{and} \qquad L^{i+1} = [L^i, \underbrace{L, \ldots, L}_{(n-1)-times}].$$

If L_1 and L_2 are Filippov algebras, then the vector space

$$L = \{(l_1, l_2) | l_1 \in L_1, l_2 \in L_2\},$$

with n-Lie bracket

$$[(l_{11}, l_{21}), \ldots, (l_{1n}, l_{2n})] = ([l_{11}, \ldots, l_{1n}], [l_{21}, \ldots, l_{2n}]),$$

is a Filippov algebra, called the *direct sum* of L_1 and L_2 and denoted by $L_1 \oplus L_2$. The following definitions are vital in our investigations.

Definition 1 Let M be an ideal of a Filippov algebra L. Then (M, L) is considered to be a pair of Filippov algebras and one may define the commutator and the center of the pair (M, L) as follws:

$$[M, \underbrace{L, \ldots, L}_{(n-1)-times}] = \langle [m, l_1, \ldots, l_{n-1}] \mid m \in M, l_i \in L \rangle,$$

and

$$Z(M, L) = \{m \in M \mid [m, l_1, \ldots, l_{n-1}] = 0 \ \forall l_i \in L\}.$$

Note that if $M = L$, then $[M, \underbrace{L, \ldots, L}_{(n-1)-times}]$ and $Z(M, L)$ are derived subalgebra and the center of L, respectively (see [18] for more information).

The lower central series for the pair (M, L) is defined as follows. First define $\gamma_1(M, L) = M$. Assume that $\gamma_c(M, L)$ is defined inductively for $c \geqslant 1$ by $\gamma_{c+1}(M, L) = [\gamma_c(M, L), \underbrace{L, \ldots, L}_{(n-1)-times}]$. Therefore, we have the series

$$M = \gamma_1(M, L) \supseteq \gamma_2(M, L) \supseteq \ldots.$$

The upper central series for the pair (M, L) is the series

$$0 = Z_0(M, L) \subseteq Z_1(M, L) = Z(M, L) \subseteq Z_2(M, L) \subseteq \ldots,$$

in which each term defined by

$$\frac{Z_{c+1}(M,L)}{Z_c(M,L)} = Z(\frac{M}{Z_c(M,L)}, \frac{L}{Z_c(M,L)}).$$

In the case $M = L$, we have the lower and upper central series of the Filippov algebra L (see [18]). One can see that

$$Z_c(M,L) \subseteq Z_c(L), \qquad \text{and} \qquad \gamma_{c+1}(M,L) \subseteq L^{c+1}.$$

Definition 2 Let (M,L) be a pair of Filippov algebras. We say that (K,H) is a subpair of (M,L), if $K \subseteq M$ and $H \subseteq L$. Also, we say that the pair (P,Q) is a quotient pair of (M,L) if P and Q are quotient Filippov algebras of M and L, respectively.

3 *c*-isoclinism between pairs of Filippov algebras

In this section, we introduce the notion of c-isoclinism for the pairs of Filippov algebras and give some equivalent conditions for pairs of Filippov algebras to be c-isoclinic.

We recall that a linear map φ from a Filippov algebra L to a Filippov algebra M is called a homomorphism if $\varphi([x_1,\dots,x_n]) = [\varphi(x_1),\dots,\varphi(x_n)]$ for any $x_1,\dots,x_n \in L$. For a pair of Filippov algebras (M,L), put $\overline{L} = L/Z_c(M,L)$ and $\overline{M} = M/Z_c(M,L)$.

Definition 3 Let (M_1,L_1) and (M_2,L_2) be two pairs of Filippov algebras and c be a non-negative integer. A c-isoclinism between (M_1,L_1) and (M_2,L_2) is a pair of isomorphisms (α,β) with $\alpha : \overline{L_1} \to \overline{L_2}$ and

$$\beta : \gamma_{c+1}(M_1,L_1) \to \gamma_{c+1}(M_2,L_2),$$

such that $\alpha(\overline{M_1}) = \overline{M_2}$ and the following diagram is commutative:

$$\begin{array}{ccc}
\overline{M_1} \oplus \underbrace{\overline{L_1} \oplus \ldots \oplus \overline{L_1}}_{c(n-1)-times} & \longrightarrow & \gamma_{c+1}(M_1,L_1) \\
\Big\downarrow {\scriptstyle \alpha^{c(n-1)+1}} & & \Big\downarrow {\scriptstyle \beta} \\
\overline{M_2} \oplus \underbrace{\overline{L_2} \oplus \ldots \oplus \overline{L_2}}_{c(n-1)-times} & \longrightarrow & \gamma_{c+1}(M_2,L_2)
\end{array}$$

In this case, we write $(M_1,L_1) \sim_c (M_2,L_2)$. We show that the map from $\overline{M_i} \oplus \underbrace{\overline{L_i} \oplus \ldots \oplus \overline{L_i}}_{c(n-1)-times}(i=1,2)$ to $\gamma_{c+1}(M_i,L_i)$ by $\gamma(c,n,M_i,L_i)(i=1,2)$. It is easy to see that $m \in M, m \in Z_c(M,L)$ if and only if

$$[\ldots[[m,l_{11},\dots,l_{1(n-1)}],l_{21},\dots,l_{2(n-1)}],\ldots],l_{c1},\dots,l_{c(n-1)}] = 0,$$

for any $l_{ij} \in L$ $(1 \leqslant i \leqslant c, 1 \leqslant j \leqslant n-1)$. Therefore, if $c \geqslant 1$, then the map $\gamma(n,c,M,L) : \dfrac{M}{Z_c(M,L)} \oplus \underbrace{\dfrac{L}{Z_c(M,L)} \oplus \ldots \oplus \dfrac{L}{Z_c(M,L)}}_{c(n-1)-times} \to \gamma_{c+1}(M,L)$ given by

$$\begin{aligned}\gamma(n,c,M,L)(m + Z_c(M,L), l_{11} + Z_c(M,L), \ldots, l_{1(n-1)} + Z_c(M,L),\\ \ldots, l_{c1} + Z_c(M,L), \ldots, l_{c(n-1)} + Z_c(M,L))\\ = [\ldots[[m, l_{11}, \ldots, l_{1(n-1)}], \ldots], l_{c1}, \ldots, l_{c(n-1)}]\end{aligned}$$

is a well-defined homomorphism.

Clearly, c-isoclinism is an equivalence relation among all pairs of Filippov algebras. If $M_1 = L_1$, $M_2 = L_2$ and $c = 1$, then a c-isoclinism between two pairs of Filippov algebras is an isoclinism between L_1 and L_2 (see [3,18] for more information).

The following results are useful in our investigations.

Lemma 1 *Let (M,L) be a pair of Filippov algebras and N be an ideal of L with $N \subseteq M$. If $N \cap \gamma_{c+1}(M,L) = 0$, then $N \subseteq Z_c(M,L)$ and $Z_c(M/N, L/N) = Z_c(M,L)/N$.*

Proof It is easy to see that for all $i \geqslant 1$,

$$(Z_i(M,L) + N)/N \subseteq Z_i(M/N, L/N).$$

Since $N \cap \gamma_{c+1}(M,L) = 0$, we have

$$\gamma_{c+1}(N,L) \subseteq N \cap \gamma_{c+1}(M,L) = 0.$$

Therefore, $N \subseteq Z_c(M,L)$ and hence $Z_c(M,L)/N \subseteq Z_c(M/N, L/N)$. Conversely, let $m + N \in Z_c(M/N, L/N)$, then for all $l_{ij} \in L(1 \leqslant i \leqslant c, 1 \leqslant j \leqslant c)$ we have

$$[\ldots[[m, l_{11}, \ldots, l_{1(n-1)}], \ldots], l_{c1}, \ldots, l_{c(n-1)}] \in N \cap \gamma_{c+1}(M,L) = 0.$$

Thus, $m \in Z_c(M,L)$ and $m + N \in Z_c(M,N)/N$. This completes the proof of the lemma.

Lemma 2 *Let (M,L) be a pair of Filippov algebras. Let N be an ideal of L with $N \subseteq M$ and H be a subalgebra of L. Then*

(i) $(H \cap M, H) \sim_c ((H \cap M) + Z_c(M,L), H + Z_c(M,L))$. *In particular, if $L = H + Z_c(M,L)$, then $(H \cap M, H) \sim_c (M,L)$. Conversely, if $H/Z_c(H \cap M, H)$ satisfies the descending chain condition for its ideals and $(H \cap M, H) \sim_c (M,L)$, then $L = H + Z_c(M,L)$.*

(ii) $(M/N, L/N) \sim_c (M/(N \cap \gamma_{c+1}(M,L)), L/(N \cap \gamma_{c+1}(M,L)))$. *In particular, if* $N \cap \gamma_{c+1}(M,L) = 0$*, then* $(M,L) \sim_c (M/N, L/N)$*. Conversely, if* $\gamma_{c+1}(M,L)$ *satisfies the ascending chain condition for its ideals and* $(M,L) \sim_c (M/N, L/N)$*, then* $N \cap \gamma_{c+1}(M,L) = 0$*.*

Proof The proof is essentially the same as in [16, Theorem 2.2] and it is omitted.

Now, by Lemma 2, we obtain the following proposition.

Proposition 1 *Let* (M,L) *be a pair of Filippov algebras and* H *be a subalgebra of* L*. If* β *is an epimorphism from* L *onto* H *such that* $\mathrm{Ker}\beta \subseteq M$*, then* β *induces a c-isoclinism between* (M,L) *and* $(\beta(M), H)$ *if and only if* $(\mathrm{Ker}\beta) \cap \gamma_{c+1}(M,L) = 0$*.*

Proof Lemma 2 (ii) gives the if part. Now, assume that β induces a c-isoclinism between (M,L) and $(\beta(M), H)$, then

$$\beta|_{\gamma_{c+1}(M,L)} : \gamma_{c+1}(M,L) \to \gamma_{c+1}(\beta(M), H),$$

is an isomorphism. Hence, $\mathrm{Ker}\beta \cap \gamma_{c+1}(M,L) = 0$.

Let (M_1, L_1) and (M_2, L_2) be two pairs of Filippov algebras. A homomorphism from (M_1, L_1) to (M_2, L_2) is a homomorphism $\varphi : L_1 \to L_2$ such that $\varphi(M_1) \subseteq M_2$. We say that (M_1, L_1) and (M_2, L_2) are isomorphic and write $(M_1, L_1) \cong (M_2, L_2)$, if φ is an isomorphism and $\varphi(M_1) = M_2$.

Now, we are able to state and prove the main results of this section.

Theorem 1 *Let* (M_1, L_1) *and* (M_2, L_2) *be two pairs of Filippov algebras and* (α, β) *be a c-isoclinism between them. Then there exists a pair* (K, H) *of Filippov algebras with subpairs* (K_1, H_1) *and* (K_2, H_2) *such that*

(i)

$$(M_1/S, L_1/S) \cong (K_1, H_1),$$

and

$$(M_2/\beta(S \cap \gamma_{c+1}(M_1, L_1)), L_2/\beta(S \cap \gamma_{c+1}(M_1, L_1)) \cong (K_2, H_2),$$

where $S = [Z_c(M_1, L_1), \underbrace{L_1, \ldots, L_1}_{(n-1)-times}]$*.*

(ii) $(K_1, H_1) \sim_c (K, H) \sim_c (K_2, H_2)$*.*

Proof Put $X = \{(l_1, l_2) \in L_1 \oplus L_2 | \alpha(l_1 + Z_c(M_1, L_1)) = l_2 + Z_c(M_2, L_2)\}$ and $Y = X \cap (M_1 \oplus M_2)$. Using the definition of isoclinism, we conclude that every generator of the subalgebra $\gamma_{c+1}(Y, X)$ is of the form $(x, \beta(x))$, where

$$x = [\ldots[[m_1, l_{11}, \ldots, l_{1(n-1)}], \ldots], l_{c1}, \ldots, l_{c(n-1)}],$$

and $m_1 \in M_1, l_{ij} \in L_1$ $(1 \leqslant i \leqslant c, 1 \leqslant j \leqslant n-1)$. In fact we have

$$\gamma_{c+1}(Y, X) = \langle (x, \beta(x)) | x \in \gamma_{c+1}(M_1, L_1) \rangle.$$

Now set

$$N_1 = \{(0, n_2) \in L_1 \oplus L_2 | n_2 \in Z_c(M_2, L_2)\},$$

and

$$N_2 = \{(n_1, 0) \in L_1 \oplus L_2 | n_1 \in Z_c(M_1, L_1)\}.$$

Then N_i is an ideal of X, $N_i \subseteq Y (i = 1, 2)$.

If $(0, n_2) = (x, y) \in \gamma_{c+1}(Y, X)$, then $x = 0, y = \beta(x) = 0$. Thus,

$$\gamma_{c+1}(Y, X) \cap N_1 = 0.$$

Also, if $(n_1, 0) = (x, y) \in \gamma_{c+1}(Y, X)$, then $y = \beta(x) = 0$ and since β is an isomorphism, we have $x = 0$. Therefore, $\gamma_{c+1}(Y, X) \cap N_2 = 0$. Put

$$L = \frac{X}{N_1} \oplus \frac{X}{\gamma_{c+1}(Y, X)}, \qquad M = \frac{Y}{N_1} \oplus \frac{Y}{\gamma_{c+1}(Y, X)}.$$

It is easy to see that $(M, L) \sim_c (M_1, L_1) \sim_c (Y, X)$. Az $N_1 \cap \gamma_{c+1}(Y, X) = 0$, X can be embedded in L by a monomorphism $i : X \to L$ defined by

$$i(x) = (x + N_1, x + \gamma_{c+1}(Y, X)).$$

Let

$$N = i(N_2) + [i(N_2), \underbrace{L, \ldots, L}_{(n-1)-times}].$$

Then N is an ideal of L. We define two homomorphism $\varphi_1 : L_1 \to L/N$ and $\varphi_2 : L_2 \to L/N$ as follows:

Let $l_1 \in L_1$ and $l_2 \in L_2$ such that $(l_1, l_2) \in X$. Define φ_1 by

$$\varphi_1(l_1) = ((l_1, l_2) + N_1, \gamma_{c+1}(Y, X)) + N.$$

Similarly, let $l_2 \in L_2$ and $l_1 \in L$ such that $(l_1, l_2) \in X$. Define φ_2 by

$$\varphi_2(l_2) = ((l_1, l_2) + N_1, (l_1, l_2) + \gamma_{c+1}(Y, X)) + N.$$

We claim that

$$\varphi_1(L_1) + Z_c(M/N, L/N) = L/N = \varphi_2(L_2) + Z_c(M/N, L/N). \qquad (1)$$

One can prove that for all $i \geqslant 1$,

$$Z_i(M_1 \oplus M_2, L_1 \oplus L_2) = Z_i(M_1, L_1) \oplus Z_i(M_2, L_2).$$

Now, lemma 1 shows that $Z_c(M, L) = \dfrac{Z_c(Y, X)}{N_1} \oplus \dfrac{X}{\gamma_{c+1}(Y, X)}$. Also, we have $(Z_c(M, L) + N)/N \subseteq Z_c(M/N, L/N)$. So, for every (l_1, l_2) and (l_1', l_2') in X,

$$\begin{aligned}
&((l_1, l_2) + N_1, (l_1', l_2') + \gamma_{c+1}(Y, X)) + N \\
&= (((l_1, l_2) + N_1, \gamma_{c+1}(Y, X)) + N) \\
&+ ((N_1, (l_1', l_2') + \gamma_{c+1}(Y, X)) + N) \\
&\in \varphi_1(L_1) + Z_c(M/N, L/N)
\end{aligned}$$

and

$$\begin{aligned}&((l_1,l_2)+N_1,(l'_1,l'_2)+\gamma_{c+1}(Y,X))+N\\&=(((l_1,l_2)+N_1,(l_1,l_2)+\gamma_{c+1}(Y,X))+N)\\&+((N_1,(l'_1-l_1,l'_2-l_2)+\gamma_{c+1}(Y,X))+N)\\&\in\varphi_2(L_2)+Z_c(M/N,L/N).\end{aligned}$$

Therefore, (1) is hold. Now we show that
(a) $\mathrm{Ker}\varphi_1=[Z_c(M_1,L_1),\underbrace{L_1,\dots,L_1}_{(n-1)-times}]$.
(b) $\mathrm{Ker}\varphi_2=\beta(\gamma_{c+1}(M_1,L_1)\cap[Z_c(M_1,L_1),\underbrace{L_1,\dots,L_1}_{(n-1)-times}])$.
(a) Let $l_1\in S=[Z_c(M_1,L_1),\underbrace{L_1,\dots,L_1}_{(n-1)-times}]$, then $(l_1,0)\in N_2$. Hence,

$$\begin{aligned}\varphi_1(l_1)&=((l_1,0)+N_1,\gamma_{c+1}(Y,X))+N\\&=(((l_1,0)+N_1,(l_1,0)+\gamma_{c+1}(Y,X))+N)\\&+((N_1,(-l_1,0)+\gamma_{c+1}(Y,X))+N)\\&=N,\end{aligned}$$

which implies that $l_1\in\mathrm{Ker}\varphi_1$. Conversely, let $l_1\in\mathrm{Ker}\varphi_1$ and choose $l_2\in L_2$ such that $(l_1,l_2)\in X$. It is clear that $((l_1,l_2)+N_1,\gamma_{c+1}(Y,X))\in N$ and for some $n_1\in Z_c(M_1,L_1)$, $c_1\in S$, we have

$$((l_1,l_2)+N_1,\gamma_{c+1}(Y,X))=$$

$$((n_1,0)+N_1,(n_1,0)+\gamma_{c+1}(Y,X))+(N_1,(c_1,0)+\gamma_{c+1}(Y,X)).$$

Hence, $l_1=n_1$ and $(n_1+c_1,0)\in\gamma_{c+1}(Y,X)$. Therefore, $n_1=-c_1\in S$ and $l_1\in S$.
(b) Suppose $l_2\in\mathrm{Ker}\varphi_2$. Choose $l_1\in L_1$ such that $(l_1,l_2)\in X$, thus

$$((l_1,l_2)+N_1,(l_1,l_2)+\gamma_{c+1}(Y,X))\in N,$$

and for some $n_1\in Z_c(M_1,L_1)$, $c_1\in S$,

$$((l_1,l_2)+N_1,(l_1,l_2)+\gamma_{c+1}(Y,X))$$

$$=((n_1,0)+N_1,(n_1+c_1,0)+\gamma_{c+1}(Y,X)).$$

Hence, $l_1=n_1$ and $(l_1-c_1-n_1,l_2)\in\gamma_{c+1}(Y,X)$, which implies that

$$l_2=\beta(l_1-c_1-n_1)\in\beta(\gamma_{c+1}(M_1,L_1)).$$

But as $c_1\in S$, we have

$$l_1-c_1-n_1=n_1-c_1-n_1\in S.$$

Therefore, $l_2 \in \beta(\gamma_{c+1}(M_1, L_1) \cap S)$. Conversely, let $l_2 \in \beta(\gamma_{c+1}(M_1, L_1) \cap S)$, then there exsits $l_1 \in \gamma_{c+1}(M_1, L_1) \cap S$ such that $l_2 = \beta(l_1)$. Therefore, similar to [16, Lemma 1.1 (i)] we have

$$\alpha(l_1 + Z_c(M_1, L_1)) = \beta(l_1) + Z_c(M_2, L_2) = l_2 + Z_c(M_2, L_2),$$

and so, $(l_1, l_2) \in X$. We know that $(l_1, l_2) \in \gamma_{c+1}(Y, X)$ and

$$\begin{aligned}((l_1, l_2) + N_1, (l_1, l_2) + \gamma_{c+1}(Y, X)) &= ((l_1, 0) + N_1, (l_1, 0) + \gamma_{c+1}(Y, X)) \\ &\quad + ((0, l_2) + N_1, (0, l_2) + \gamma_{c+1}(Y, X)) \\ &= ((l_1, 0) + N_1, (l_1, 0) + \gamma_{c+1}(Y, X)) \\ &\quad + (N_1, (-l_1, 0) + \gamma_{c+1}(Y, X)) \\ &\in N\end{aligned}$$

So, $\varphi_2(l_2) = 0$.
Now, set $(K, H) = (M/N, L/N)$ and $(K_i, H_i) = (\varphi_i(M_i), \varphi_i(L_i)), (i = 1, 2)$. Then

$$(M_1/S, L_1/S) \cong (K_1, H_1),$$

$$(M_2/\beta(S \cap \gamma_{c+1}(M_1, L_1)), L_2/\beta(S \cap \gamma_{c+1}(M_1, L_1)) \cong (K_2, H_2),$$

and

$$(K_1, H_1) \sim_c (K, H) \sim_c (K_2, H_2).$$

In the following theorem, we show that two pairs (M_1, L_1) and (M_2, L_2) of Filippov algebras are c-isoclinic if and only if there exists a pair (M, L) of Filippov algebras such that (M_1, L_1) and (M_2, L_2) occur as quotient pairs of (M, L), also (M, L), (M_1, L_1) and (M_2, L_2) are c-isoclinic to each other.

Theorem 2 *Let (M_1, L_1) and (M_2, L_2) be two pairs of Filippov algebras. Then $(M_1, L_1) \sim_c (M_2, L_2)$ if and only if there exists a pair of Filippov algebras (M, L) and there exist ideals N_1 and N_2 of L with $N_1 \subseteq M, N_2 \subseteq M$ such that*

$$(M_1, L_1) \cong (M/N_1, L/N_1), (M/N_2, L/N_2) \cong (M_2, L_2),$$

and

$$(M_1, L_1) \sim_c (M, L) \sim_c (M_2, L_2).$$

Proof Clearly, the necessity of the conditions hold. Conversely, suppose that $(M_1, L_1) \sim_c (M_2, L_2)$ and (α, β) be an isoclinism between them. Let L be a subalgebra of $L_1 \oplus L_2$ given by

$$L = \{(l_1, l_2) \in L_1 \oplus L_2 | \alpha(l_1 + Z_c(M, L)) = l_2 + Z_c(M, L)\},$$

and $M = L \cap (M_1 \oplus M_2)$. Assume $N_1 = \{(0, n_2) \in L_1 \oplus L_2 | n_2 \in Z_c(M_2, L_2)\}$ and $N_2 = \{(n_1, 0) \in L_1 \oplus L_2) | n_1 \in Z_c(M_1, L_1)\}$. Then N_1 and N_2 are ideals of L such that $N_i \subseteq M$ and $(M_i, L_i) \cong (M/N_i, L/N_i)$, $(i = 1, 2)$. Also, we have

$$\gamma_{c+1}(M, L) = \langle (x, \beta(x)) | x \in \gamma_{c+1}(M_1, L_1) \rangle.$$

Similar to the proof of Theorem 1,

$$\gamma_{c+1}(M,L) \cap N_1 = 0, \qquad \text{and} \qquad \gamma_{c+1}(M,L) \cap N_2 = 0.$$

Now, by Lemma 2 (ii)

$$(M/N_1, L/N_1) \sim_c (M,L) \sim_c (M/N_2, L/N_2), \ \textit{as required.}$$

Let (M_1, L_1) and (M_2, L_2) be two pairs of Filippov algebras. If $M_1 = L_1$ and $M_2 = L_2$, then a c-isoclinism between them is a c-isoclinism between L_1 and L_2. Now, by the definition of c-isoclinism for Filippov algebras we have the following results.

Lemma 3 *Let L be a Filippov algebras. Then $L \sim_c L \oplus A$ for each Filippov algebra A with $A^{c+1} = 0$.*

Proof Clearly, for all $i \geqslant 1$, $(L \oplus A)^i = L^i \oplus A^i$. Therefore, we have

$$(L \oplus A)^{c+1} = L^{c+1}, \qquad \text{and} \qquad Z_c(L \oplus A) = Z_c(L) \oplus A.$$

Now, define $\alpha : L/Z_c(L) \to (L \oplus A)/Z_c(L \oplus A)$ by $\alpha(x + Z_c(L)) = x + Z_c(L) \oplus A$. Let β be a identity map on L^{c+1}. It is easy to see that (α, β) is a c-isoclinism from L to $L \oplus A$.

Proposition 2 *Let L_1 and L_2 be two Filippov algebras. Then $L_1 \sim_c L_2$ if and only if there exists a Filippov algebra L and there exist ideals N_1 and N_2 of L such that*

$$\frac{L}{N_1} \sim_c \frac{L}{N_1} \oplus \frac{L}{L^{c+1}} \sim_c H_1,$$
$$\frac{L}{N_2} \sim_c \frac{L}{N_2} \oplus \frac{L}{L^{c+1}} \sim_c H_2,$$

and $H_1 \cong L \cong H_2$, for some subalgebra H_1 of $\frac{L}{N_1} \oplus \frac{L}{L^{c+1}}$ and some subalgebra H_2 of $\frac{L}{N_2} \oplus \frac{L}{L^{c+1}}$, where L, N_1 and N_2 are defined in the proof of Theorem 2.

Proof We have $M = L$, $\gamma_{c+1}(M,L) = L^{c+1}$ and $Z_c(M,L) = Z_c(L)$. Put

$$H_1 = \{(l + N_1, l + L^{c+1}) | l \in L\}.$$

Clearly, H_1 is a subalgebra of $\frac{L}{N_1} \oplus \frac{L}{L^{c+1}}$ and the map $\delta : L \to H_1$ given by $\delta(l) = (l + N_1, l + L^{c+1})$ is an isomorphism. Now, we show that

$$\frac{L}{N_1} \oplus \frac{L}{L^{c+1}} \sim_c H_1.$$

By Lemma 2(i), it is enough to prove that

$$\frac{L}{N_1} \oplus \frac{L}{L^{c+1}} = H_1 + Z_c\left(\frac{L}{N_1} \oplus \frac{L}{L^{c+1}}\right).$$

Suppose $(l_1 + N_1, l_2 + L^{c+1}) \in \frac{L}{N_1} \oplus \frac{L}{L^{c+1}}$. It is obvious that

$$\begin{aligned}(l_1 + N_1, l_2 + L^{c+1}) &= (l_1 + N_1, l_1 + L^{c+1}) \\ &+ (N_1, (l_2 - l_1) + L^{c+1}) \\ &\in H_1 + Z_c(\frac{L}{N_1} \oplus \frac{L}{L^{c+1}}).\end{aligned}$$

The reverse containment is clear and the result obtains. By a similar argument,

$$L \cong H_2 \sim_c \frac{L}{N_2} \oplus \frac{L}{L^{c+1}},$$

in which

$$H_2 = \{(l + N_2, l + L^{c+1}) | l \in L\},$$

and H_2 is a subalgebra of $\frac{L}{N_2} \oplus \frac{L}{L^{c+1}}$.

In the next proposition, we show that any two isoclinic Filippov algebras L_1 and L_2 can be realized as subalebras of the Filippov algebras $\widehat{L}$, where $\widehat{L}$, L_1 and L_2 are isoclinic to each other.

Proposition 3 *Let L_1 and L_2 be two pairs of Filippov algebras. Then $L_1 \sim_c L_2$ if and only if there exists a Filippov alfebra $\widehat{L}$ containing subalgebra $\widehat{H_1}, \widehat{H_2}$ such that the following statements hold:*

(i) $L_1 \cong \widehat{H_1} \sim_c \widehat{L} \sim_c \widehat{H_2} \cong L_2$.

(ii) $\frac{L}{N_1} \sim_c \frac{L}{N_1} \oplus \frac{L}{L^{c+1}} \sim_c \widehat{L} \sim_c \widehat{H_2}$, *where L, N_1 are defined in the proof of Theorem 2.*

Proof Set $V = \frac{L}{N_1} \oplus \frac{L}{L^{c+1}}$ and

$$W = \{((l, 0) + N_1, (l, 0) + L^{c+1} | l \in Z_c(L_1) \cap L_1^c\}.$$

It is easy to see that W is an ideal of V such that $W \cap V^{c+1} = 0$.
Let $l_1 \in L_1$ and choose $l_2 \in L_2$ such that $(l_1, l_2) \in L$ and define $\delta_1 : L_1 \to V/W$ by

$$\delta_1(l_1) = ((l_1, l_2) + N_1, L^{c+1}) + W.$$

Similarly, let $l_2 \in L_2$ and choose $l_1 \in L_1$ such that $(l_1, l_2) \in L$, define

$$\delta_2 : L_2 \to V/W,$$

by

$$\delta_2(l_2) = ((l_1, l_2) + N_1, (l_1, l_2) + L^{c+1}) + W.$$

We claim that δ_1 and δ_2 are monomorphism and

$$\delta_1(L_1) + Z_c(V/W) = V/W = \delta_2(L_2) + Z_c(V/W).$$

To prove these claims, take $l_1 \in L_1$ with $\delta_1(l_1) = 0$. Then for some

$$l' \in Z_c(L_1) \cap L_1^c, \qquad \text{and} \qquad l_2 \in \alpha(l_1 + Z_c(L)),$$

$(l_1 - l', l_2) \in N_1$, where $(l', 0) \in L^{c+1}$. Therefore, $l_1 = l' = 0$. Thus, $\mathrm{Ker}\delta_1 = 0$. By a similar argument, we obtain $\mathrm{Ker}\delta_2 = 0$. Also, for each $(l_1, l_2), (l_1', l_2') \in L$, we have

$$\begin{aligned}
&((l_1, l_2) + N_1, (l_1', l_2') + L^{c+1}) + W \\
&= (((l_1, l_2) + N_1, L^{c+1}) + W) \\
&+ ((N_1, (l_1', l_2') + L^{c+1}) + W) \\
&\in \delta_1(L_1) + Z_c(V/W),
\end{aligned}$$

and

$$\begin{aligned}
&((l_1, l_2) + N_1, (l_1', l_2') + L^{c+1}) + W \\
&= (((l_1, l_2) + N_1, (l_1, l_2) + L^{c+1}) + W) \\
&+ ((N_1, (l_1 - l_1', l_2 - l_2') + L^{c+1}) + W) \\
&\in \delta_2(L_2) + Z_c(V/W).
\end{aligned}$$

Now, putting $\widehat{L} = V/W, \delta_1(L_1) = \widehat{H_1}$, and $\delta_2(L_2) = \widehat{H_2}$, the results follow.

The following corollary is an immediate consequence of the above results.

Corollary 1 *Let L_1 and L_2 be two Filippov algebras. Then the following statements are equivalent:*

(i) L_1 and L_2 are c-isoclinic.

(i) There exists a Filippov algebra A, a subalgebra P of $L_1 \oplus A$ with $P + Z_c(L_1 \oplus A) = L_1 \oplus A$ and an ideal Z of P with $Z \cap P^{c+1} = 0$ such that $P/Z \cong L_2$.

(ii) There exists a Filippov algebra B, an ideal Z of $L_1 \oplus B$ with $Z \cap (L_1 \oplus B)^{c+1} = 0$, and a subalgebra P of $\dfrac{L_1 \oplus B}{Z}$ with

$$P + Z_c\left(\frac{L_1 \oplus B}{Z}\right) = \frac{L_1 \oplus B}{Z},$$

such that P is isomorphic to L_2.

Proof Suppose that $L_1 \sim_c L_2$, then (i) follows by taking

$$A = L/L^{c+1}, P = H_1,$$

and $Z = N_2$ in Proposition 2. We obtain part (ii) by taking

$$B = L/L^{c+1}, \quad Z = W,$$

and $P = \widehat{H_2}$ in Proposition 3.

4 c-perfect pairs of Filippov algebras

This section is devoted to study c-perfect pairs of Filippov algebras.

Definition 4 A pair (M, L) of Filippov algebras is said to be c-perfect if $\gamma_{c+1}(M, L) = M$.

Clearly, a Filippov algebra L is a c-perfect Filippov algebra, if $L^{c+1} = L$.

The following theorems give the connections between c-perfect and c-isoclinic pairs of Filippov algebras.

Theorem 3 *Let L be a finite dimensional c-perfect Filippov algebra with $Z_c(L) = 0$. Then any c-isoclinic Filippov algebra H to L is isomorphic to the direct sum of L by $Z_c(H)$.*

Proof By the assumption,

$$H \sim_c L = L/Z_c(L) \cong H/Z_c(H) \text{ and } H \sim_c L = L^{c+1} \cong H^{c+1}.$$

By Lemma 2, $Z_c(H) \cap H^{c+1} = 0$ and $H = H^{c+1} + Z_c(H)$. Therefore,

$$H = \gamma_{c+1}(H) \oplus Z_c(H).$$

Theorem 4 *Let (M, L) be a pair of finite dimensional Filippov algebras. If $(H \cap M, H)$ is a c-perfect pair of Filippov algebras such that H is a subalgebra of L and $(M, L) \sim_c (H \cap M, H)$, then*

$$L = \gamma_{c+1}(M, L) + Z_c(M, L) + K,$$

for some subalgebra K of H. In particular, if $H \subseteq M$, then

$$L = \gamma_{c+1}(M, L) + Z_c(M, L).$$

Proof By Lemma 2, we have $L = H + Z_c(M, L)$ and by the assumption $\gamma_{c+1}(H \cap M, H) = H \cap M$. Let K be a subalgebra of H such that

$$K + H \cap M = H,$$

then $L = \gamma_{c+1}(M, L) + Z_c(M, L) + K$.

Theorem 5 *Let (M, L) be a pair of finite dimensional Filippov algebras and (N, H) be a pair of finite dimensional Filippov algebras such that*

$$(N, H) \sim_c (M, L),$$

and $\dim M = \dim N$. *If (M, L) is c-perfect or $Z_c(M, L) = 0$, then $M \cong N$.*

Proof By the definition of isoclinism, we have the following isomorphisms

$$\alpha : L/Z_c(M, L) \to H/Z_c(N, H), \alpha(M/Z_c(M, L)) = N/Z_c(N, H),$$
$$\beta : \gamma_{c+1}(M, L) \to \gamma_{c+1}(N, H).$$

Now, clearly if $\gamma_{c+1}(M, L) = M$, then $\dim M = \dim \gamma_{c+1}(N, H)$.
Since, $\dim M = \dim N$, we have $\gamma_{c+1}(N, H) = N$ and hence $N \cong M$. If $Z_c(M, L) = 0$, then the result follows immediately.

Theorem 6 *Let (M, L) be a c-perfect pair of finite dimensional Filippov algebras, then L contains no proper subalgebra H such that $(H \cap M, H) \sim_c (M, L)$.*

Proof Assume that (M, L) is a c-perfect pair with a subalgebra H of L such that $(H \cap M, H) \sim_c (M, L)$, then by Lemma 2, $L = H + Z_c(M, L)$. Also, $\gamma_{c+1}(H \cap M, H) \cong \gamma_{c+1}(M, L) = M$ and hence, $\gamma_{c+1}(H \cap M, H) = M \subseteq H \cap M$. It implies that $M \subseteq H$. Let K be a complement of M in H as vector spaces, then

$$L = M \oplus K + Z_c(M, L) = M \oplus K = H.$$

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