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On Schur Multiplier of a Pair of Lie Algebras

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Abstract Nilpotent Lie algebras have played an important role in mathematics in the classification theory of Lie algebras. Let (N, L) be a pair of finite dimensional Lie algebras. Let K be an ideal of L such that $L = N \oplus K$ and N be a filiform ideal of L. Also, let dim $N = n$ and $\dim K = m$. Then $s'(N, L) = \frac{1}{2}(n-1)(n-2) + 1 + (n-1)m - \dim \mathcal{M}(N, L)$. In this paper, we characterize the pair (N, L) for $s'(N, L) = 0, 1, 2, \ldots, 15$.

Keywords Filiform Lie algebras · Pair of Lie algebras · Schur multiplier

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1 Introduction

Let (N, L) be a pair of Lie algebras, in which N is an ideal in L. If N admits a complement in L, then the Schur multiplier of the pair (N, L), $\mathcal{M}(N, L)$ is defined to be the factor Lie algebra $(R \cap [S, F])/[R, F]$, in which S is an ideal in F such that $N \cong S/R$ (see [2], fore more information). In particular, if $N = L$, then $\mathcal{M}(L, L) = \mathcal{M}(L)$ is the Schur multiplier of L (see [3,6, 9]). Moneyhun [9] proved that if L is a Lie algebra of dimension n, then $\dim \mathcal{M}(L) = \frac{1}{2}n(n-1) - t(L)$, where $t(L)$ is a non-negative integer. In [3,5,6], all nilpotent Lie algebras are characterized, when $t(L) = 0, 1, \ldots, 8$. Let (N, L) be a pair of finite dimensional nilpotent Lie algebras. Saeedi et al. [13] proved that if N admits a complement K say, in L with dim $N = n$ and dim $K = m$,

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then

$$\dim \mathcal{M}(N, L) = \frac{1}{2}n(n + 2m - 1) - t(N, L), \quad (1)$$

where $t(N, L) \geq 0$. This gives us the Moneyhun's result, if $m = 0$. The author and colleagues [2] characterized the pair (N, L), for which $t(N, L) = 0, 1, 2, 3, 4$. Moreover, they determined pairs (N, L) for $t(N, L) = 0, 1, \ldots, 10$, when L is a filiform Lie algebra. Also, Niroomand and Russo [11] proved that

$$\dim \mathcal{M}(L) \leq \frac{1}{2}(n + m - 2)(n - m - 1) + 1, \quad (2)$$

where L is a non-abelian nilpotent Lie algebra with dim $L = n$ and dim $L^2 = m$. The above upper bound implies that dim $\mathcal{M}(L) = \frac{1}{2}(n-1)(n-2)+1-s(L)$, where $s(L) \geq 0$. Niroomand et al. in [10–12] classified the structure of L, when $s(L) = 0, 1, 2, 3$. Moreover, it is proved under some conditions that

$$\dim \mathcal{M}(N, L) = \frac{1}{2}(n - 1)(n - 2) + 1 + (n - 1)m - s'(N, L), \quad (3)$$

where $s'(N, L) \geq 0$, $\dim N = n$ and $\dim K = m$.

In the present paper, we characterize all pairs (N, L) when N is a filiform Lie algebra and $s'(N, L) = 0, 1, 2, \ldots, 15$. Note that in the proof of main theorem, the upper bound (2) enables us to provide a new technique in our classification which makes our upper bound smaller than the one in (1).

2 Main Results

In this section, first we discuss some results which will be used in the main theorem. A Lie algebra L is filiform if L has maximal nilpotency class(see [2] for more information). We recall that a Lie algebra L is called a Heisenberg algebra provided that $L^2 = Z(L)$ and $\dim L^2 = 1$. A Heisenberg Lie algebra has odd dimension with a basis $e, e_1, \ldots, e_{2m}$ subject to the relations $[e_{2i-1}, e_{2i}] = e$ for $i = 1, \ldots, m$. The Heisenberg Lie algebra of dimension $2m + 1$ is denoted by $H(m)$. A Lie algebra L is abelian, if $[x, y] = 0$, for all $x, y \in L$ and $A(n)$ will denote the abelian Lie algebra of dimension n. In Theorem 1, we extend Theorem 5 of [1]. These results are similar to the work of B. Mashayekhy etal. in the case of groups (2013). See [[8], Theorems 2.1 and 2.2].

The following lemma plays an essential role in our investigations.

Lemma 1 *([14], Theorem 2.3) Let L be a non-abelian n-dimensional nilpotent Lie algebra of maximal class and $n \geq 4$. Then* $0 \leq s(L) \leq 15$ *if and only if L is isomorphic to one of the Lie algebras* $L_{4,3}$, $L_{5,6}$ $L_{5,7}$, $L_{6,15}$, $L_{6,16}$, $L_{6,17}$, $L_{6,18}$, L_1, L_2, L_3, L_4, L_5, L_6, L_7, *or* L_8.

Theorem 1 *Let (N, L) be a pair of finite dimensional Lie algebras such that L is a nilpotent Lie algebra, N is a non-abelian n-dimensional nilpotent Lie algebra of maximal class and $n \geq 4$. Also, let K be an ideal of L such that* $L = N \oplus K$, $\dim N = n$, $\dim K = m$ *and*

$$s' = s'(N,L) = \frac{1}{2}(n-1)(n-2) + 1 + (n-1)m - \dim \mathcal{M}(N,L).$$

Then

1. *In the cases* $s' = 0, 1$ *there are no pairs.*
2. $s' = 2$ *if and only if* $(N,L) \cong (L_{4,3}, L_{4,3})$.
3. $s' = 3$ *if and only if* $(N,L) \cong (L_{4,3}, L_{4,3} \oplus A(1))$.
4. $s' = 4$ *if and only if* (N,L) *is isomorphic to one of the following pairs:*
$$(L_{5,6}, L_{5,6}), (L_{5,7}, L_{5,7}) \text{ or } (L_{4,3}, L_{4,3} \oplus A(2)).$$
5. $s' = 5$ *if and only if* $(N,L) \cong (L_{4,3}, L_{4,3} \oplus A(3))$.
6. $s' = 6$ *if and only if* (N,L) *is isomorphic to one of the following pairs:*
$$(L_{4,3}, L_{4,3} \oplus A(4)), (L_{5,6}, L_{5,6} \oplus A(1)), \text{ or } (L_{5,7}, L_{5,7} \oplus A(1)).$$
7. $s' = 7$ *if and only if* (N,L) *is isomorphic to one of the following pairs:*
$$(L_{4,3}, L_{4,3} \oplus A(5)) \text{ or } (L_{4,3}, L_{4,3} \oplus H(1)).$$
8. $s' = 8$ *if and only if* (N,L) *is isomorphic to one of the following pairs:*
$(L_{4,3}, L_{4,3} \oplus A(6))$, $(L_{4,3}, L_{4,3} \oplus H(1) \oplus A(1))$,
$(L_{5,6}, L_{5,6} \oplus A(2))$, $(L_{5,7}, L_{5,7} \oplus A(2))$,
$(L_{6,15}, L_{6,15})$, $(L_{6,17}, L_{6,17})$,
$(L_{6,18}, L_{6,18})$.
9. $s' = 9$ *if and only if* (N,L) *is isomorphic to one of the following pairs:*
$(L_{4,3}, L_{4,3} \oplus A(7))$, $(L_{4,3}, L_{4,3} \oplus H(1) \oplus A(2))$,
$(L_{4,3}, L_{4,3} \oplus H(2))$, $(L_{6,14}, L_{6,14})$,
$(L_{6,16}, L_{6,16})$.
10. $s' = 10$ *if and only if* (N,L) *is isomorphic to one of the following pairs:*
$(L_{4,3}, L_{4,3} \oplus H(1) \oplus A(3))$, $(L_{4,3}, L_{4,3} \oplus H(2) \oplus A(1))$,
$(L_{4,3}, L_{4,3} \oplus A(8))$, $(L_{5,6}, L_{5,6} \oplus A(3))$,
$(L_{5,7}, L_{5,7} \oplus A(3))$.
11. $s' = 11$ *if and only if* (N,L) *is isomorphic to one of the following pairs:*
$(L_{4,3}, L_{4,3} \oplus H(1) \oplus A(4))$, $(L_{4,3}, L_{4,3} \oplus H(2) \oplus A(2))$,
$(L_{4,3}, L_{4,3} \oplus A(9))$, $(L_{4,3}, L_{4,3} \oplus L(4,5,2,4))$,
$(L_{4,3}, L_{4,3} \oplus L(4,5,1,6))$, $(L_{6,15}, L_{6,15} \oplus A(1))$,
$(L_{6,17}, L_{6,17} \oplus A(1))$, $(L_{6,18}, L_{6,18} \oplus A(1))$.
12. $s' = 12$ *if and only if* (N,L) *is isomorphic to one of the following pairs:*
$(L_{4,3}, L_{4,3} \oplus H(1) \oplus A(5))$, $(L_{4,3}, L_{4,3} \oplus H(2) \oplus A(3))$,
$(L_{4,3}, L_{4,3} \oplus A(10))$, $(L_{4,3}, L_{4,3} \oplus L(4,5,2,4) \oplus A(1))$,
$(L_{4,3}, L_{4,3} \oplus L(4,5,1,6) \oplus A(1))$, $(L_{4,3}, L_{4,3} \oplus L(5,6,2,7))$,
$(L_{4,3}, L_{4,3} \oplus L'(5,6,2,7))$, $(L_{4,3}, L_{4,3} \oplus L(7,6,2,7))$,
$(L_{4,3}, L_{4,3} \oplus L(7,6,2,7,\beta_1,\beta_2))$, $(L_{5,6}, L_{5,6} \oplus H(1))$,
$(L_{5,6}, L_{5,6} \oplus A(4))$, $(L_{5,7}, L_{5,7} \oplus H(1))$,
$(L_{5,7}, L_{5,7} \oplus A(4))$, $(L_{6,14}, L_{6,14} \oplus A(1))$,
$(L_{6,16}, L_{6,16} \oplus A(1))$, (L_1, L_1),
(L_8, L_8) *for* $\lambda = 3$, (L_2, L_2),
(L_4, L_4), (L_5, L_5).

13. $s' = 13$ *if and only if* (N, L) *is isomorphic to one of the following pairs:*

$(L_{4,3}, L_{4,3} \oplus H(1) \oplus A(6))$, $(L_{4,3}, L_{4,3} \oplus H(2) \oplus A(4))$,
$(L_{4,3}, L_{4,3} \oplus A(11))$, $(L_{4,3}, L_{4,3} \oplus L'(7,5,1,7))$,
$(L_{4,3}, L_{4,3} \oplus L(5,6,2,7) \oplus A(1))$, $(L_{4,3}, L_{4,3} \oplus L'(5,6,2,7) \oplus A(1))$,
$(L_{4,3}, L_{4,3} \oplus L(7,6,2,7) \oplus A(1))$, $(L_{4,3}, L_{4,3} \oplus L(7,6,2,7,\beta_1,\beta_2) \oplus A(1))$,
$(L_{4,3}, L_{4,3} \oplus L(4,5,2,4) \oplus A(2))$, $(L_{4,3}, L_{4,3} \oplus L(4,5,1,6) \oplus A(2))$,
$(L_{4,3}, L_{4,3} \oplus L(7,5,2,7))$, $(L_{4,3}, L_{4,3} \oplus L(7,5,1,7))$,
(L_3, L_3), (L_6, L_6),
(L_7, L_7), (L_8, L_8) *for* $\lambda \neq 3$.

14. $s' = 14$ *if and only if* (N, L) *is isomorphic to one of the following pairs:*

$(L_{4,3}, L_{4,3} \oplus A(12))$, $(L_{4,3}, L_{4,3} \oplus H(1) \oplus A(7))$,
$(L_{4,3}, L_{4,3} \oplus H(2) \oplus A(5))$, $(L_{4,3}, L_{4,3} \oplus L(4,5,2,4) \oplus A(3))$,
$(L_{4,3}, L_{4,3} \oplus L(4,5,1,6) \oplus A(3))$, $(L_{4,3}, L_{4,3} \oplus L(5,6,2,7) \oplus A(2))$,
$(L_{4,3}, L_{4,3} \oplus L'(5,6,2,7) \oplus A(2))$, $(L_{4,3}, L_{4,3} \oplus L(7,6,2,7) \oplus A(2))$,
$(L_{4,3}, L_{4,3} \oplus L(7,6,2,7,\beta_1,\beta_2) \oplus A(2))$, $(L_{4,3}, L_{4,3} \oplus L(7,5,2,7) \oplus A(1))$,
$(L_{4,3}, L_{4,3} \oplus L(7,5,1,7) \oplus A(1))$, $(L_{4,3}, L_{4,3} \oplus L'(7,5,1,7) \oplus A(1))$,
$(L_{5,6}, L_{5,6} \oplus H(1) \oplus A(1))$, $(L_{5,6}, L_{5,6} \oplus A(5))$,
$(L_{5,7}, L_{5,7} \oplus H(1) \oplus A(1))$, $(L_{5,7}, L_{5,7} \oplus A(5))$,
$(L_{6,15}, L_{6,15} \oplus A(2))$, $(L_{6,17}, L_{6,17} \oplus A(2))$,
$(L_{6,18}, L_{6,18} \oplus A(2))$.

15. $s' = 15$ *if and only if* (N, L) *is isomorphic to one of the following pairs:*

$(L_{4,3}, L_{4,3} \oplus A(13))$, $(L_{4,3}, L_{4,3} \oplus H(1) \oplus A(8))$,
$(L_{4,3}, L_{4,3} \oplus H(2) \oplus A(6))$, $(L_{4,3}, L_{4,3} \oplus L(4,5,2,4) \oplus A(4))$,
$(L_{4,3}, L_{4,3} \oplus L(4,5,1,6) \oplus A(4))$, $(L_{4,3}, L_{4,3} \oplus L(5,6,2,7) \oplus A(3))$,
$(L_{4,3}, L_{4,3} \oplus L'(5,6,2,7) \oplus A(3))$, $(L_{4,3}, L_{4,3} \oplus L(7,6,2,7) \oplus A(3))$,
$(L_{4,3}, L_{4,3} \oplus L(7,6,2,7,\beta_1,\beta_2) \oplus A(3))$, $(L_{4,3}, L_{4,3} \oplus L(7,5,2,7) \oplus A(2))$,
$(L_{4,3}, L_{4,3} \oplus L(7,5,1,7) \oplus A(2))$, $(L_{4,3}, L_{4,3} \oplus L'(7,5,1,7) \oplus A(2))$,
$(L_{6,14}, L_{6,14} \oplus A(2))$, $(L_{6,16}, L_{6,16} \oplus A(2))$.

Proof The necessity of theorem follows from $L = N \oplus K$ and Lemma 1.4 of [7]. For sufficiency, put $s = s(N) = \frac{1}{2}(n-1)(n-2) + 1 - \dim \mathcal{M}(N)$. Thus, by Lemma 1.4 of [7], we have

$$mn - m = (s' - s) + (\dim N/N^2)(\dim K/K^2). \tag{4}$$

Hence, $s \leq s'$. Now, suppose that $s' = 0$, then $s = 0$. So, there are no pairs by Lemma 1. If $s' = 1$, then $s = 0, 1$ and so, there are no pairs by Lemma 1.

Assume that $s' = 2$. If $s = 2$, then by Lemma 1, $N \cong L_{4,3}$. Hence, by (4) we have $m = 0$, which implies that $(N, L) \cong (L_{4,3}, L_{4,3})$.

case $s' = 3$. If $s = 2$, then by Lemma 1, $N \cong L_{4,3}$ and $m = 1$. This implies that $K \cong A(1)$ and so, $(N, L) \cong (L_{4,3}, L_{4,3} \oplus A(1))$.

case $s' = 4$. If $s = 2$, then $N \cong L_{4,3}$. Thus, by Lemma 1, $m = 2$ and so, $(N, L) \cong (L_{4,3}, L_{4,3} \oplus A(2))$. If $s = 4$, then by (4), $N \cong L_{5,6}\, or\, L_{5,7}$. If $N \cong L_{5,6}$, then by (4), $m = 0$ and so, $(N, L) \cong (L_{5,6}, L_{5,6})$. Assume that $N \cong L_{5,7}$, then $m = 0$ and hence, $(N, L) \cong (L_{5,7}, L_{5,7})$.

case $s' = 5$. If $s = 2$, then $m = 3$ and $\dim K^2 = 0$. Thus, $(N, L) \cong (L_{4,3}, L_{4,3} \oplus A(3))$. If $s = 4$, then, $N \cong L_{5,6}\, or\, L_{5,7}$. Hence, there are no pairs by (4).

case $s' = 6$. If $s = 2$, then $N \cong L_{4,3}$ and so, by (4) we have $(N, L) \cong (L_{4,3}, L_{4,3} \oplus A(4))$. If $s = 4$, then $N \cong L_{5,6}\, or\, L_{5,7}$. and so, by (4) we have

$$(N, L) \cong (L_{5,6}, L_{5,6} \oplus A(1))\, or\, (L_{5,7}, L_{5,7} \oplus A(1)).$$

case $s' = 7$. If $s = 2$, then by Lemma (1) $N \cong L_{4,3}$. Thus, by (4) we obtain

$$(N, L) \cong (L_{4,3}, L_{4,3} \oplus A(5))\, or\, (L_{4,3}, L_{4,3} \oplus H(1)).$$

If $s = 4$, then there are no pairs by Lemma 1 and (4).

case $s' = 8$. If $s = 2$, then, by Lemma 1 and (4),

$$(N, L) \cong (L_{4,3}, L_{4,3} \oplus A(6))\, or\, (L_{4,3}, L_{4,3} \oplus H(1) \oplus A(1)).$$

If $s = 4$, then

$$(N, L) \cong (L_{5,6}, L_{5,6} \oplus A(2))\, or\, (L_{5,7}, L_{5,7} \oplus A(2)).$$

Assume that $s = 8$, then by Lemma 1, $N \cong L_{6,15}, L_{6,17}\, or\, L_{6,18}$. Hence by (4) we have

$$(N, L) \cong (L_{6,15}, L_{6,15}), (L_{6,17}, L_{6,17}),\, or\, (L_{6,18}, L_{6,18}).$$

case $s' = 9$. If $s = 2$, then by Lemma 1, $N \cong L_{4,3}$ and so, by (4) we have

$$(N, L) \cong (L_{4,3}, L_{4,3} \oplus A(7)), (L_{4,3}, L_{4,3} \oplus H(1) \oplus A(2))\ or\ (L_{4,3}, L_{4,3} \oplus H(2)).$$

If $s = 4$, then by Lemma 1, we have $N \cong L_{5,6}\ or\ L_{5,7}$ and so, there are no pairs by (4). If $s = 8$, then there are no pairs by Lemma 1 and (4). If $s = 9$, then by Lemma 1 and (4) we have

$$(N, L) \cong (L_{6,14}, L_{6,14})\ or\ (L_{6,16}, L_{6,16}).$$

case $s' = 10$. If $s = 2$, then $N \cong L_{4,3}$ and so, by (4) we have

$$(N, L) \cong (L_{4,3}, L_{4,3} \oplus H(1) \oplus A(3)), (L_{4,3}, L_{4,3} \oplus A(8)) \; or \; (L_{4,3}, L_{4,3} \oplus H(2) \oplus A(1)).$$

If $s = 4$, then by Lemma 1 and (4) we have

$$(N, L) \cong (L_{5,6}, L_{5,6} \oplus A(3)) \; or \; (L_{5,7}, L_{5,7} \oplus A(3)).$$

If $s = 8$, then there are no pairs by Lemma 1 and (4). If $s = 9$, similarly there are no pairs.

case $s' = 11$. If $s = 2$, then by Lemma 1 and (4), we have

$$(N, L) \cong (L_{4,3}, L_{4,3} \oplus H(1) \oplus A(4)), (L_{4,3}, L_{4,3} \oplus H(2) \oplus A(2)) \; or \; (L_{4,3}, L_{4,3} \oplus A(9)).$$

If $s = 4$, then there are no pairs by Lemma 1 and (4). Assume that $s = 8$. Then by Lemma 1 and (4) we have

$$\begin{array}{ll} (N, L) \cong (L_{6,15}, L_{6,15} \oplus A(1)), & (L_{6,17}, L_{6,17} \oplus A(1), \\ (L_{6,18}, L_{6,18} \oplus A(1)), & (L_{4,3}, L_{4,3} \oplus L(4,5,2,4)), \\ (L_{4,3}, L_{4,3} \oplus L(4,5,1,6)). & \end{array}$$

If $s = 9$, then by Lemma 1 and (4), there are no pairs.

case $s' = 12$. If $s = 2$, then by Lemma 1 and (4) we have

$$(N, L) \cong (L_{4,3}, L_{4,3} \oplus H(1) \oplus A(5)), (L_{4,3}, L_{4,3} \oplus H(2) \oplus A(3)) \; or \; (L_{4,3}, L_{4,3} \oplus A(10)).$$

If $s = 4$, then by Lemma 1 and (4) we have

$$(N, L) \cong (L_{5,6}, L_{5,6} \oplus H(1)), (L_{5,6}, L_{5,6} \oplus A(4)), (L_{5,7}, L_{5,7} \oplus H(1)) \; or \; (L_{5,7}, L_{5,7} \oplus A(4)).$$

If $s = 8$, then by Lemma 1 and (4), there are no pairs. If $s = 9$, then by Lemma 1 and (4) we have

$$\begin{array}{ll} (N, L) \cong (L_{6,14}, L_{6,14} \oplus A(1)), & (L_{6,16}, L_{6,16} \oplus A(1)), \\ (L_{4,3}, L_{4,3} \oplus L(4,5,1,6) \oplus A(1)), & (L_{4,3}, L_{4,3} \oplus L(5,6,2,7)), \\ (L_{4,3}, L_{4,3} \oplus L'(5,6,2,7)), & (L_{4,3}, L_{4,3} \oplus L(7,6,2,7)), \\ (L_{4,3}, L_{4,3} \oplus L(7,6,2,7,\beta_1,\beta_2)). & \end{array}$$

If $s = 12$, then by Lemma 1 and (4), we have

$$(N, L) \cong (L_1, L_1), (L_2, L_2), (L_4, L_4), (L_5, L_5) \, or \, (L_8, L_8) \;\; for \; \lambda = 3.$$

case $s' = 13$. If $s = 2$, then by Lemma 1 and (4) we have

$$\begin{array}{ll} (N, L) \cong (L_{4,3}, L_{4,3} \oplus H(1) \oplus A(6)), & (L_{4,3}, L_{4,3} \oplus H(2) \oplus A(4)), \\ (L_{4,3}, L_{4,3} \oplus A(11)), & (L_{4,3}, L_{4,3} \oplus L'(7,5,1,7)), \\ (L_{4,3}, L_{4,3} \oplus L(5,6,2,7) \oplus A(1)), & (L_{4,3}, L_{4,3} \oplus L'(5,6,2,7) \oplus A(1)), \\ (L_{4,3}, L_{4,3} \oplus L(7,6,2,7) \oplus A(1)), & (L_{4,3}, L_{4,3} \oplus L(7,6,2,7,\beta_1,\beta_2) \oplus A(1)), \\ (L_{4,3}, L_{4,3} \oplus L(4,5,2,4) \oplus A(2)), & (L_{4,3}, L_{4,3} \oplus L(4,5,1,6) \oplus A(2)), \\ (L_{4,3}, L_{4,3} \oplus L(7,5,2,7)), & (L_{4,3}, L_{4,3} \oplus L(7,5,1,7)), \end{array}$$

If $s = 4, 8, 9, 12$, then there are no pairs by (4) and Lemma 1. If $s = 13$, then we can see that

$$(N, L) \cong (L_3, L_3), (L_6, L_6), (L_7, L_7)\, or\, (L_8, L_8)\ for\ \lambda \neq 3.$$

case $s' = 14$. Similar to the previous cases we have:

$(N, L) \cong (L_{4,3}, L_{4,3} \oplus A(12))$,
$(L_{4,3}, L_{4,3} \oplus H(1) \oplus A(7))$,
$(L_{4,3}, L_{4,3} \oplus H(2) \oplus A(5))$,
$(L_{4,3}, L_{4,3} \oplus L(4, 5, 2, 4) \oplus A(3))$,
$(L_{4,3}, L_{4,3} \oplus L(4, 5, 1, 6) \oplus A(3))$,
$(L_{4,3}, L_{4,3} \oplus L(5, 6, 2, 7) \oplus A(2))$,
$(L_{4,3}, L_{4,3} \oplus L'(5, 6, 2, 7) \oplus A(2))$,
$(L_{4,3}, L_{4,3} \oplus L(7, 6, 2, 7) \oplus A(2))$,
$(L_{4,3}, L_{4,3} \oplus L(7, 6, 2, 7, \beta_1, \beta_2) \oplus A(2))$,
$(L_{4,3}, L_{4,3} \oplus L(7, 5, 2, 7) \oplus A(1))$,
$(L_{4,3}, L_{4,3} \oplus L(7, 5, 1, 7) \oplus A(1))$,
$(L_{4,3}, L_{4,3} \oplus L'(7, 5, 1, 7) \oplus A(1))$,
$(L_{5,6}, L_{5,6} \oplus H(1) \oplus A(1))$,
$(L_{5,6}, L_{5,6} \oplus A(5))$,
$(L_{5,7}, L_{5,7} \oplus H(1) \oplus A(1))$,
$(L_{5,7}, L_{5,7} \oplus A(5))$,
$(L_{6,15}, L_{6,15} \oplus A(2))$,
$(L_{6,17}, L_{6,17} \oplus A(2))$,
$(L_{6,18}, L_{6,18} \oplus A(2))$.

case $s' = 15$. Similar to the previous cases we have:

$(N, L) \cong (L_{4,3}, L_{4,3} \oplus A(13))$,
$(L_{4,3}, L_{4,3} \oplus H(1) \oplus A(8))$,
$(L_{4,3}, L_{4,3} \oplus H(2) \oplus A(6))$,
$(L_{4,3}, L_{4,3} \oplus L(4, 5, 2, 4) \oplus A(4))$,
$(L_{4,3}, L_{4,3} \oplus L(4, 5, 1, 6) \oplus A(4))$,
$(L_{4,3}, L_{4,3} \oplus L(5, 6, 2, 7) \oplus A(3))$,
$(L_{4,3}, L_{4,3} \oplus L'(5, 6, 2, 7) \oplus A(3))$,
$(L_{4,3}, L_{4,3} \oplus L(7, 6, 2, 7) \oplus A(3))$,
$(L_{4,3}, L_{4,3} \oplus L(7, 6, 2, 7, \beta_1, \beta_2) \oplus A(3))$,
$(L_{4,3}, L_{4,3} \oplus L(7, 5, 2, 7) \oplus A(2))$,
$(L_{4,3}, L_{4,3} \oplus L(7, 5, 1, 7) \oplus A(2))$,
$(L_{4,3}, L_{4,3} \oplus L'(7, 5, 1, 7) \oplus A(2))$,
$(L_{6,14}, L_{6,14} \oplus A(2))$,
$(L_{6,16}, L_{6,16} \oplus A(2))$.

Here $H(m)$ denotes the Hiesenberg Lie algebra of dimension $2m + 1$, $A(m)$ is an m-dimensional abelian Lie algebra and $L(a, b, c, d)$ denotes the Lie algebra discovered for the case $t(L) = a$, where $b = \dim L$, $c = \dim Z(L)$ and $d = t(L)$. (See [3, 5, 6] for more information).

Table 1

dim L	Non Zero Multiplication	Nilpotent Lie algebra
3	$[x_1, x_2] = x_3$	$H(1)$
4	$[x_1, x_2] = x_3$	$H(1) \oplus A(1)$
5	$[x_1, x_2] = x_3$	$H(1) \oplus A(2)$
4	$[x_1, x_2] = x_3, [x_1, x_3] = x_4$	$L(3, 4, 1, 4) = L_{4,3}$
5	$[x_1, x_2] = x_3, [x_1, x_4] = x_5$	$L(4, 5, 2, 4)$
6	$[x_1, x_2] = x_3$	$H(1) \oplus A(3)$
5	$[x_1, x_2] = x_5, [x_3, x_4] = x_5$	$H(2)$
7	$[x_1, x_2] = x_3$	$H(1) \oplus A(4)$
5	$[x_1, x_2] = x_3, [x_1, x_3] = x_5$	$L(3, 4, 1, 4) \oplus A(1)$
5	$[x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5$	$L(4, 5, 1, 6)$
6	$[x_1, x_2] = x_5, [x_1, x_3] = x_5, [x_3, x_4] = x_5$	$H(2) \oplus A(1)$
6	$[x_1, x_2] = x_3, [x_1, x_4] = x_6$	$L(4, 5, 2, 4) \oplus A(1)$
8	$[x_1, x_2] = x_3$	$H(1) \oplus A(5)$
5	$[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5$	$L(7, 5, 2, 7)$
5	$[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5$	$L(7, 5, 1, 7)$
5	$[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5$	$L'(7, 5, 1, 7)$
6	$[x_1, x_2] = x_3, [x_1, x_4] = x_6, [x_2, x_5] = x_6$	$L(5, 6, 2, 7)$
6	$[x_1, x_2] = x_3, [x_4, x_5] = x_6$	$L'(5, 6, 2, 7)$
6	$[x_1, x_2] = x_5, [x_3, x_4] = x_6$	$L(7, 6, 2, 7)$
6	$[x_1, x_2] = x_5 + \beta_1 x_6, [x_3, x_4] = x_5, [x_1, x_4] = x_6, [x_3, x_2] = \beta_2 x_6$	$L(7, 6, 2, 7, \beta_1, \beta_2)$
7	$[x_1, x_2] = x_5, [x_3, x_4] = x_5$	$H(2) \oplus A(2)$
7	$[x_1, x_2] = x_7, [x_3, x_4] = x_7, [x_5, x_6] = x_7$	$H(3)$
6	$[x_1, x_2] = x_3, [x_1, x_3] = x_6$	$L(3, 4, 1, 4) \oplus A(2)$
6	$[x_1, x_2] = x_3, [x_1, x_3] = x_6, [x_2, x_4] = x_6$	$L(4, 5, 1, 6) \oplus A(1)$
7	$[x_1, x_2] = x_3, [x_1, x_4] = x_7$	$L(4, 5, 2, 4) \oplus A(2)$
8	$[x_1, x_2] = x_5, [x_3, x_4] = x_5$	$H(2) \oplus A(3)$
8	$[x_1, x_2] = x_7, [x_3, x_4] = x_7, [x_5, x_6] = x_7$	$H(3) \oplus A(1)$
10	$[x_1, x_2] = x_3$	$H(1) \oplus A(7)$

Table 2 Seven-dimensional nilpotent Lie algebras of maximal class

Name	Nonzero multiplication	dim $\mathcal{M}(L)$
$L_1 = (123457A)$	$[x_1, x_i] = x_{i+1}, 2 \leq i \leq 6$	4
$L_1 = (123457B)$	$[x_1, x_i] = x_{i+1}, 2 \leq i \leq 6$	
	$[x_2, x_3] = x_7$	4
$L_3 = (123457C)$	$[x_1, x_i] = x_{i+1}, 2 \leq i \leq 6$	
	$[x_2, x_5] = x_7, [x_3, x_4] = -x_7$	3
$L_4 = (123457D)$	$[x_1, x_i] = x_{i+1}, 2 \leq i \leq 6$	
	$[x_2, x_4] = x_7, [x_2, x_3] = x_6$	4
$L_5 = (123457E)$	$[x_1, x_i] = x_{i+1}, 2 \leq i \leq 6$	
	$[x_2, x_4] = x_7, [x_2, x_3] = x_6 + x_7$	4
$L_6 = (123457F)$	$[x_1, x_i] = x_{i+1}, 2 \leq i \leq 6$	
	$[x_3, x_4] = -x_7, [x_2, x_3] = x_6$	
	$[x_2, x_4] = [x_2, x_5] = x_7$	3
$L_7 = (123457H)$	$[x_1, x_i] = x_{i+1}, 2 \leq i \leq 6$	
	$[x_2, x_4] = x_6, [x_2, x_5] = x_7$	3
	$[x_2, x_3] = x_5 + x_7$	
$L_8 = (123457I)$	$[x_1, x_i] = x_{i+1}, 2 \leq i \leq 6$	
	$[x_2, x_5] = \lambda x_7, [x_3, x_4] = (1 - \lambda) x_7$	
	$[x_2, x_3] = x_5, [x_2, x_4] = x_6$	4 for $\lambda = 3$, and 3 otherwise

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Table 3 Lie algebras of maximal class of dimension n for $4 \leq n \leq 6$.

Name	Basis	Nonzero multiplication	$\dim \mathcal{M}(L)$
$L_{4,3}$	$x_1, \dots, x_4$	$[x_1, x_2] = x_3, [x_1, x_3] = x_4$	2
$L_{5,7}$	$x_1, \dots, x_5$	$[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5$	3
$L_{5,6}$	$x_1, \dots, x_5$	$[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5$ $[x_2, x_3] = x_5$	3
$L_{6,14}$	$x_1, \dots, x_6$	$[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5$ $[x_2, x_5] = [x_4, x_3] = x_6, [x_1, x_4] = x_5$	2
$L_{6,15}$	$x_1, \dots, x_6$	$[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5$ $[x_1, x_5] = [x_2, x_4] = x_6, [x_2, x_3] = x_5$	3
$L_{6,16}$	$x_1, \dots, x_6$	$[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5$ $[x_2, x_5] = [x_4, x_3] = x_6,$	2
$L_{6,17}$	$x_1, \dots, x_6$	$[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5$ $[x_1, x_5] = [x_2, x_3] = x_6,$	3
$L_{6,18}$	$x_1, \dots, x_5, x_6$	$[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5$ $[x_1, x_5] = x_6$	3

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