Global Analysis and Discrete Mathematics
Volume 7, Issue 2, pp. 193–208
ISSN: 2476-5341 *Research Article*

Isoclinisms in n-Hom-Lie Algebras

Mina Sadeghloo · Mahboubeh Alizadeh Sanati

Received: 8 November 2022 / Accepted: 1 March 2023

Abstract In the present article, we define the concept of isoclinism for n-Hom-Lie algebras and investigate some of its properties. Also, we introduce the factor sets on n-Hom-Lie algebras. As a result, it is shown that the equivalency between isoclinism and isomorphism of two finite-dimensional n-Hom-Lie algebras just depends on whether one of them be regular.

Keywords Isoclinism · n-Hom-Lie algebra · Factor set

Mathematics Subject Classification (2010) 17B61 · 17B99 · 17B66

1 Introduction

In 2002, Hartwig, Larsson, and Silvestrov introduced the notion of Hom-Lie algebras [15], and gave some of their fundamental properties which are studied in mathematical physics, for generalizing the Yang-Barter equation and to braid group representations [28], [29]. A Hom-Lie algebra is an F-vector space equipped with the bilinear skew-symmetric bracket which satisfies a Jacobi identity twisted by a linear map φ. When φ is identity, then the definition of Hom-Lie algebras coincides on Lie algebras. The construction of Hom-Lie algebras as a generalization of Lie algebras, induces the category of Hom-Lie algebras which is denoted by HomLie [5]. In fact, Hom-Lie algebras can be considered as the objects of the category HomLie, and its morphisms are Lie algebra homomorphisms $f : (V, \varphi) \longrightarrow (W, \psi)$ such that $f \circ \varphi = \psi \circ f$. Hom-Lie algebras are studied in several concepts of Lie algebras such as semisimple Lie

M. Sadeghloo (Corresponding Author)
Mathematical Institute, Golestan University, Gorgan, Iran.
Tel.: +98-1732254163
E-mail: mina.sadeghloo@yahoo.com

M. Alizadeh Sanati
Mathematical Institute, Golestan University, Gorgan, Iran.
E-mail: m.alizadeh@gu.ac.ir

algebras, (co)homology theory, representation, universal central extensions, non-abelian tensor products, and simple Lie algebras, respectively in [16, 27, 1, 9, 25, 7, 8, 18].

Philip Hall, in 1940, introduced group isoclinism [14], and Kay Moneyhun extended this notion to Lie algebras in 1994 and defined factor sets for Lie algebras. As a result, it was shown that for a given finite dimension, isomorphism and isoclinism are equivalent [19].

The concept of n-Lie algebras was defined by Filippov in 1987. Also, he proved all n-Lie algebras of dimension $n+1$ over an algebraically closed field were classified [13]. Eshrati and Moghaddam presented similar results of isoclinism in n-Lie algebras. Utilizing the notion of isoclinism, they proved that isomorphism and isoclinism are identical on n-Lie algebras of the same finite dimension [12]. The notion of an n-Hom-Lie algebra which is a generalization of an n-Lie algebra was introduced in 2011, [2]. Then several aspects of algebraic structures about n-Hom-Lie algebras, for example, the cohomologies, central extensions and deformations were studied.

The first purpose of this paper, is to provide a definition of isoclinism for an n-Hom-Lie algebra. We investigate several properties about isoclinism of n-Hom-Lie algebras. By defining Hom-stem n-Hom-Lie algebras, we prove that the Hom-centers of two isoclinic Hom-stem n-Hom-Lie algebras are isomorphic. Finally, we introduce the notion of factor sets on n-Hom-Lie algebras. As a conclusive result, we show that the equivalency between isoclinism and isomorphism of two finite-dimensional n-Hom-Lie algebras depend on that only one of them be regular.

2 Preliminaries

Throughout this paper, we fix F as a ground field and all the vector spaces are considered over F and linear maps are F-linear maps.

Definition 1 A Lie algebra $(V,[-,-])$ with a linear map $\varphi : V \longrightarrow V$ is called *Hom-Lie algebra* provided

(i) $[x,y]=-[y,x]$, (skew-symmetry)
(ii) $[\varphi(x),[y,z]]+[\varphi(y),[z,x]]+[\varphi(z),[x,y]]=0$, (Hom-Jacobi identity)

for all $x,y,z \in V$.
In this paper, we assume that φ preserves the product which is called *multiplicative*, i.e. $\varphi([x,y])=[\varphi(x),\varphi(y)]$, for all $x,y \in V$.

In the case of $\varphi = id_V$, Hom-Lie algebras are exactly Lie algebras. A vector space V endowed with a linear map

$$\varphi : V \longrightarrow V,$$

is called *Hom-vector space.* A Hom-vector space (V,φ) with the trivial bracket and any linear map $\varphi : V \longrightarrow V$ constructs a Hom-Lie algebra which is called *abelian* Hom-Lie algebra.

Example 1 For any Lie algebra V and Lie algebra endomorphism

$$\varphi : V \longrightarrow V,$$

we have Hom-Lie algebra (V, φ), if we define the bracket by $[x, y] := [\varphi(x), \varphi(y)]$, for all x, y in V.

Definition 2 A *Hom-Lie subalgebra* of (V, φ) is a vector subspace W of V, which is closed under bracket and φ, i.e. $[w, w'], \varphi(w) \in W$, for all $w, w' \in W$. A Hom-Lie subalgebra $(W, \varphi_|)$ which $\varphi_|$ is restriction of φ to W, is said to be an *ideal* if $[w, v] \in W$, for all $w \in W, v \in V$. A Hom-Lie algebra (V, φ) is said to be *regular* if φ is bijective. The set

$$Z_\varphi(V) = \{x \in V | \; [\varphi^k(x), v] = 0, \quad \forall v \in V, k \geq 0\},$$

where $\varphi^0 = id_V$ and φ^k, $k \geq 1$ is the k times composition of φ with itself, is the largest central ideal of (V, φ) which is called the *Hom-center* of (V, φ). If φ is surjective or (V, φ) is abelian, then $Z_\varphi(V)$ equals to

$$Z(V)(= \{x \in V | \; [x, v] = 0, \quad \forall v \in V\}).$$

Let (V, φ) and (W, ψ) be two Hom-Lie algebras. A linear map $f : V \longrightarrow W$ is a *Hom-Lie algebra morphism*, if $f([v_1, v_2]) = [f(v_1), f(v_2)]$, for all $v_1, v_2 \in V$ and $f \circ \varphi = \psi \circ f$. This property may be more palatable by asserting that the following diagram is commutative.

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \varphi \downarrow & & \downarrow \psi \\ V & \xrightarrow{f} & W \end{array}$$

A generalization of Hom-Lie algebras is the following notion which is defined by H. Ataguema, A. Makhlouf, and S. Silvestrov in [3].

Definition 3 An *n-Hom-Lie algebra* is a vector space V equipped with a bracket operation

$$[-, \ldots, -] : \underbrace{V \oplus \cdots \oplus V}_{n-times} \to V,$$

and a linear map $\varphi : V \to V$ such that for all $v_1, \ldots, v_n, w_2, \ldots, w_n \in V$ the following identity holds

$$\Big[[v_1, \ldots, v_n], \varphi(w_2), \ldots, \varphi(w_n)\Big] = \sum_{i=1}^{n} \Big[\varphi(v_1), \ldots, [v_i, w_2, \ldots, w_n], \varphi(v_{i+1}), \ldots, \varphi(v_n)\Big].$$

A subspace W of n-Hom-Lie algebra (V, φ), which is closed under the n-Lie bracket and invariant by φ is called *n-Hom-Lie subalgebra*. An n-Hom-Lie subalgebra W of V is called an *n-Hom-Lie ideal*, provided

$$[W, \underbrace{V, \ldots, V}_{(n-1)-times}] \subseteq W.$$

The n-Hom-Lie ideal generated by $\langle [v_1, \ldots, v_n] \mid v_i \in V \rangle$ is the *derived* ideal and denoted by V^2. The *Hom-center* of n-Hom-Lie algebra (V, φ) is defined as

$$Z_\varphi(V) = \{x \in V : [\varphi^k(x), v_1, \ldots, v_{n-1}] = 0 \;\; \forall\, v_i \in V, 1 \leq i \leq n-1, k \geq 0\},$$

which is an ideal.

In our investigation the following definition is fundamental.

Definition 4 Let (V, φ) and (W, ψ) be two n-Hom-Lie algebras and

$$\alpha : V/Z_\varphi(V) \longrightarrow W/Z_\psi(W),$$

and $\beta : V^2 \longrightarrow W^2$ be two Hom-Lie algebra morphisms such that the following diagram commutes

$$\begin{array}{ccc} V/Z_\varphi(V) \oplus \cdots \oplus V/Z_\varphi(V) & \xrightarrow{\rho} & V^2 \\ {\scriptstyle \alpha^n}\big\downarrow & & \big\downarrow{\scriptstyle \beta} \\ W/\psi(W) \oplus \cdots \oplus W/\psi(W) & \xrightarrow{\sigma} & W^2 \end{array}$$

in which ρ and σ are defined by $\rho(\overline{v}_1, \ldots, \overline{v}_n) = [v_1, \ldots, v_n]$, for all

$$\overline{v}_i = v_i + Z_\varphi(V) \in V/Z_\varphi(V),$$

and $\sigma(\widetilde{w}_1, \ldots, \widetilde{w}_n) = [w_1, \ldots, w_n]$, for all

$$\widetilde{w}_i = w_i + Z_\psi(W) \in W/Z_\psi(W), 1 \leqslant i \leqslant n.$$

In other words, $\beta([v_1, \ldots, v_n]) = [w_1, \ldots, w_n]$, whenever $w_i \in \alpha(v_i + Z_\varphi(V))$ for $i = 1, \ldots, n$. Then the pair (α, β) is called *homoclinism* and if they are both isomorphism, then (α, β) is *isoclinism* and we write $V \sim W$.

The proof of the following lemmas are straightforward, so we refer the reader to [4] for obtaining more information.

Lemma 1 *If (V, φ) is an n-Hom-Lie algebra and (W, ψ) is an abelian n-Hom-Lie algebra, then $V \cong V \oplus W$.*

Lemma 2 *Let N be an ideal of n-Hom-Lie algebra (V, φ). Then we have*

(i) $N \cap V^2 = 0$ implies $V \sim V/N$.
(ii) If (V, φ) is of finite dimension and $V \sim V/N$, then $N \cap V^2 = 0$.

Lemma 3 *If (α, β) is the isoclinism pair between two n-Hom-Lie algebras (V, φ) and (W, ψ), then*

(i) $\alpha(a + Z_\varphi(V)) = \beta(a) + Z_\psi(W)$.
(ii) $\beta([a, v_2, \ldots, v_n]) = [\beta(a), w_1, \ldots, w_n]$, *for all*

$$a \in V^2, \quad v_i \in V, \quad w_i \in \alpha(v_i + Z_\varphi(V)), \quad 2 \leq i \leq n.$$

In 1994, Moneyhun defined the notion of stem Lie algebra, [19]. Now, we define *Hom-stem* n-Hom-Lie algebras which some results in the next section are given based on this concept. An n-Hom-Lie algebra $(V, [-, \ldots, -], \varphi)$ is called *Hom-stem* if $Z_\varphi(V) \subseteq V^2$.
The existance of a Hom-stem n-Hom-Lie algebra in each isoclinism family of n-Hom-Lie algebras, is stated in the following lemma which can be proved easily.

Lemma 4 *Let $\mathcal{L}$ be an isoclinism family of n-Hom-Lie algebras. Then*
(i) $\mathcal{L}$ contains a Hom-stem n-Hom-Lie algebra.
(ii) Any finite-dimensional n-Hom-Lie algebra (V, φ) in $\mathcal{L}$ is Hom-stem if and only if (V, φ) has a minimal dimension in $\mathcal{L}$.

Lemma 5 *Let (V, φ) be an n-Hom-Lie algebra and $V = U \oplus Z_\varphi(V)$ and φ be injective, then $\varphi(U) \subseteq U$.*

The following proposition shows that the Hom-centers of two isoclinic Hom-stem n-Hom-Lie algebras are isomorphic.

Proposition 1 *If (V, φ) and (W, ψ) are two isoclinic Hom-stem n-Hom-Lie algebras, then $Z_\varphi(V) \cong Z_\psi(W)$.*

Proof Let (α, β) be an isoclinism pair between (V, φ) and (W, ψ). Since $Z_\varphi(V) \subseteq V^2$, by using Lemma 3 (i),

$$\alpha(v + Z_\varphi(V)) = \beta(v) + Z_\psi(W),$$

for all $v \in Z_\varphi(V)$, which implies $\beta(v) \in Z_\psi(W)$ and thus $\beta(Z_\varphi(V)) \subseteq Z_\psi(W)$. On the other hand, for $z \in Z_\psi(W)$ there exists $x \in V$ such that $\alpha(x + Z_\varphi(V)) = z + Z_\psi(W) = 0$. Now, by Lemma 3 (ii), we have

$$0 = [\psi^k(z), w_1, \ldots, w_n] = [\psi^k(\beta(x)), v_1, \ldots, v_n],$$

where $\beta(x) = z \in Z_\psi(W)$ and $w_i \in \alpha(v_i + Z_\varphi(V))$. Hence $Z_\psi(W) = \beta(Z_\varphi(V))$ and consequently $Z_\varphi(V) \cong Z_\psi(W)$.

3 Factor sets in n-Hom-Lie algebras

In studying n-Hom-Lie algebras, the concept of factor sets is a basic tool. In 1994, the factor sets in Lie algebras are defined by Moneyhun, [19]. In this section, we introduce them for n-Hom-Lie algebras and investigate some of their properties.

Definition 5 Let $(V,[-,\ldots,-],\varphi)$ be a finite-dimensional n-Hom-Lie algebra. The n-bilinear map

$$r : \frac{V}{Z_\varphi(V)} \oplus \cdots \oplus \frac{V}{Z_\varphi(V)} \longrightarrow Z_\varphi(V),$$

is said to be a *factor set* when

(i) $[\overline{v}_1,\ldots,\overline{v}_i,\ldots,\overline{v}_j,\ldots,\overline{v}_n]=0$, for all $\overline{v}_k \in V/Z_\varphi(V)$ with $\overline{v}_i=\overline{v}_j$,

(ii) $$r\Big([\overline{v}_1,\ldots,\overline{v}_n],\widetilde{\varphi}\,(\overline{w}_2),\ldots,\widetilde{\varphi}\,(\overline{w}_n)\Big) = \sum_{i=1}^{n} r(\widetilde{\varphi}\,(\overline{v}_1),\ldots,[\overline{v}_i,\overline{w}_2,\ldots,\overline{w}_n],\ldots,\widetilde{\varphi}\,(\overline{v}_n)),$$

for all $\overline{v}_i,\overline{w}_j \in V/Z_\varphi(V), 1\le i\le n$, $2\leqslant j\leqslant n$, where

$$\widetilde{\varphi}\colon V/Z_\varphi(V) \longrightarrow V/Z_\varphi(V),$$

defined by $\widetilde{\varphi}\,(\overline{v}) := \varphi(v)+Z_\varphi(V), \quad \forall\, \overline{v}\in V/Z_\varphi(V)$. The factor set r is said to be *multiplicative* if

$$r(\widetilde{\varphi}\,(\overline{v}_1),\ldots,\widetilde{\varphi}\,(\overline{v}_n)) = \varphi r(\overline{v}_1,\ldots,\overline{v}_n), \quad \forall\, \overline{v}_i \in V/Z_\varphi(V), (1\le i\le n).$$

Lemma 6 *Let $(V,[-,\ldots,-],\varphi)$ be an n-Hom-Lie algebra and r be a factor set on $(V,[-,\ldots,-],\varphi)$. Define*

$$R = (Z_\varphi(V), \tfrac{V}{Z_\varphi(V)}, r) = \Big\{(a,\overline{v}) : a\in Z_\varphi(V), \overline{v}\in \tfrac{V}{Z_\varphi(V)}\Big\}.$$

Then

(i) (R,ψ) is an n-Hom-Lie algebra with an n-bilinear map defined by

$$[(a_1,\overline{v}_1),\ldots,(a_n,\overline{v}_n)] := (r(\overline{v}_1,\ldots,\overline{v}_n),[\overline{v}_1,\ldots,\overline{v}_n]), \quad (1)$$

for all $(a_1,\overline{v}_1),\ldots,(a_n,\overline{v}_n)\in R$ and the linear map $\psi : R\longrightarrow R$ is given by

$$\psi((a,\overline{v})) := (\varphi(a),\widetilde{\varphi}\,(\overline{v})), \quad \forall (a,\overline{v})\in R. \quad (2)$$

(ii) If r is multiplicative, then (R,ψ) is regular.

(iii) $Z_R := \{(a,0)\in R : a\in Z_\varphi(V)\} \cong Z_\varphi(V)$.

Proof (i) We need to check only the properties of being n-Hom-Lie algebra. Clearly, the first identity holds. To check the Hom-Jacobi identity, we have

$$\Big[[(a_1,\overline{v}_1),\dots,(a_n,\overline{v}_n)],\psi(b_2,\overline{w}_2),\dots,\psi(b_n,\overline{w}_n)\Big] \overset{(1),(2)}{=}$$

$$\Big[(r(\overline{v}_1,\dots,\overline{v}_n),[\overline{v}_1,\dots,\overline{v}_n]),(\varphi(b_2),\widetilde{\varphi}\,(\overline{w}_2)),\dots,(\varphi(b_n),\widetilde{\varphi}\,(\overline{w}_n))\Big] \overset{(1)}{=}$$

$$\Big(r([\overline{v}_1,\dots,\overline{v}_n],\widetilde{\varphi}\,(\overline{w}_2),\dots,\widetilde{\varphi}\,(\overline{w}_n)),[[\overline{v}_1,\dots,\overline{v}_n],\widetilde{\varphi}\,(\overline{w}_2),\dots,\widetilde{\varphi}\,(\overline{w}_n)]\Big) =$$

$$\Big(\sum_{i=1}^{n} r(\widetilde{\varphi}\,(\overline{v}_1),\dots,\widetilde{\varphi}\,(\overline{v}_{i-1}),[\overline{v}_i,\overline{w}_2,\dots,\overline{w}_n],\widetilde{\varphi}\,(\overline{v}_{i+1}),\dots,\widetilde{\varphi}\,(\overline{v}_n)),$$

$$\sum_{i=1}^{n}[\widetilde{\varphi}\,(\overline{v}_1),\dots,\widetilde{\varphi}\,(\overline{v}_{i-1}),[\overline{v}_i,\overline{w}_2,\dots,\overline{w}_n],\widetilde{\varphi}\,(\overline{v}_{i+1}),\dots,\widetilde{\varphi}\,(\overline{v}_n)]\Big) =$$

$$\sum_{i=1}^{n}\Big(r(\widetilde{\varphi}\,(\overline{v}_1),\dots,\widetilde{\varphi}\,(\overline{v}_{i-1}),[\overline{v}_i,\overline{w}_2,\dots,\overline{w}_n],\widetilde{\varphi}\,(\overline{v}_{i+1}),\dots,\widetilde{\varphi}\,(\overline{v}_n)),$$

$$[\widetilde{\varphi}\,(\overline{v}_1),\dots,\widetilde{\varphi}\,(\overline{v}_{i-1}),[\overline{v}_i,\overline{w}_2,\dots,\overline{w}_n],\widetilde{\varphi}\,(\overline{v}_{i+1}),\dots,\widetilde{\varphi}\,(\overline{v}_n)]\Big) \overset{(1)}{=}$$

$$\sum_{i=1}^{n}\Big[(\varphi(a_1),\widetilde{\varphi}\,(\overline{v}_1)),\dots,(\varphi(a_{i-1}),\widetilde{\varphi}\,(\overline{v}_{i-1})),[(a_i,\overline{v}_i),(b_2,\overline{w}_2),\dots,(b_n,\overline{w}_n)],$$

$$\dots,(\varphi(a_{i+1}),\widetilde{\varphi}\,(\overline{v}_{i+1})),\dots,(\varphi(a_n),\widetilde{\varphi}\,(\overline{v}_n))\Big] \overset{(2)}{=}$$

$$\sum_{i=1}^{n}\Big[\psi(a_1,\overline{v}_1),\dots,\psi(a_{i-1},\overline{v}_{i-1}),[(a_i,\overline{v}_i),(b_2,\overline{w}_2),\dots,(b_n,\overline{w}_n)],$$

$$\dots,\psi(a_{i+1},\overline{v}_{i+1}),\dots,\psi(a_n,\overline{v}_n)\Big],$$

for all $(a_i,\overline{v}_i),(b_i,\overline{w}_i)\in R$. Thus (R,ψ) is an n-Hom-Lie algebra.
The proofs of (ii) and (iii) are obvious.

The following lemma proves the existence of the factor set for a given n-Hom-Lie algebra and gives the connection between them.

Lemma 7 *For an n-Hom-Lie algebra $(V,[-,\dots,-],\varphi)$, there exists a factor set r such that*

$$V \cong \Big(Z_\varphi(V),\frac{V}{Z_\varphi(V)},r\Big).$$

Proof Let K be a complement of $Z_\varphi(V)$ in V, i.e. $V=K\oplus Z_\varphi(V)$. Now, we define the map $\theta: V/Z_\varphi(V)\longrightarrow V$ such that

$$\theta(\overline{v})=\theta(v+Z_\varphi(V))=\theta(k+a+Z_\varphi(V))=k,$$

when $v\in V, a\in Z_\varphi(V), k\in K$. Clearly, $\overline{\theta(\overline{v})}=\overline{v}$ and so

$$[\theta(\overline{v}_1),\dots,\theta(\overline{v}_n)]-\theta[\overline{v}_1,\dots,\overline{v}_n]\in Z_\varphi(V),$$

for all $\overline{v}_1, \dots, \overline{v}_n \in V/Z_\varphi(V)$. Now, define

$$r : \frac{V}{Z_\varphi(V)} \oplus \cdots \oplus \frac{V}{Z_\varphi(V)} \longrightarrow Z_\varphi(V)$$

given by

$$r(\overline{v}_1, \dots, \overline{v}_n) := [\theta(\overline{v}_1), \dots, \theta(\overline{v}_n)] - \theta[\overline{v}_1, \dots, \overline{v}_n].$$

First, we have $\theta \widetilde{\varphi} = \varphi\theta$, because

$$\theta \widetilde{\varphi} (\overline{v}) = \theta \widetilde{\varphi} (k + a + Z_\varphi(V)) = \theta(\varphi(k) + Z_\varphi(V)) = \varphi(k),$$

and

$$\varphi\theta(\overline{v}) = \varphi(\theta(k + a + Z_\varphi(V))) = \varphi(k),$$

for all $\overline{v} = \overline{k + a} \in V/Z_\varphi(V)$, where $k \in K, a \in Z_\varphi(V)$.
To show that r is a factor set, we only need to check the Hom-Jacobi identity as follows.

$$\begin{aligned}
& r\Big([\overline{v}_1, \dots, \overline{v}_n], \widetilde{\varphi} (\overline{w}_2), \dots, \widetilde{\varphi} (\overline{w}_n)\Big) = \\
& \Big[\theta[\overline{v}_1, \dots, \overline{v}_n], \theta(\widetilde{\varphi} (\overline{w}_2)), \dots, \theta(\widetilde{\varphi} (\overline{w}_n))\Big] - \theta\Big[[\overline{v}_1, \dots, \overline{v}_n], \widetilde{\varphi} (\overline{w}_2), \dots, \widetilde{\varphi} (\overline{w}_n)\Big] = \\
& \Big[[\theta(\overline{v}_1), \dots, \theta(\overline{v}_n)] + z, \varphi\theta(\overline{w}_2), \dots, \varphi\theta(\overline{w}_n)\Big] - \theta\Big[[\overline{v}_1, \dots, \overline{v}_n], \widetilde{\varphi} (\overline{w}_2), \dots, \widetilde{\varphi} (\overline{w}_n)\Big] = \\
& \sum_{i=1}^{n} [\varphi\theta(\overline{v}_1), \dots, \varphi\theta(\overline{v}_{i-1}), [\theta(\overline{v}_i), \theta(\overline{w}_2), \dots, \theta(\overline{w}_n)], \varphi\theta(\overline{v}_{i+1}), \dots, \varphi\theta(\overline{v}_n)] \\
& - \theta\Big(\sum_{i=1}^{n} [\widetilde{\varphi} (\overline{v}_1), \dots, \widetilde{\varphi} (\overline{v}_{i-1}), [\overline{v}_i, \overline{w}_2, \dots, \overline{w}_n], \widetilde{\varphi} (\overline{v}_{i+1}), \dots, \widetilde{\varphi} (\overline{v}_n)]\Big) = \\
& \sum_{i=1}^{n} \Big([\theta \widetilde{\varphi} (\overline{v}_1), \dots, \theta \widetilde{\varphi} (\overline{v}_{i-1}), \theta([\overline{v}_i, \overline{w}_2, \dots, \overline{w}_n]), \theta \widetilde{\varphi} (\overline{v}_{i+1}), \dots, \theta \widetilde{\varphi} (\overline{v}_n)] \\
& - \theta([\widetilde{\varphi} (\overline{v}_1), \dots, \widetilde{\varphi} (\overline{v}_{i-1}), [\overline{v}_i, \overline{w}_2, \dots, \overline{w}_n], \widetilde{\varphi} (\overline{v}_{i+1}), \dots, \widetilde{\varphi} (\overline{v}_n)])\Big) = \\
& \sum_{i=1}^{n} r(\widetilde{\varphi} (\overline{v}_1), \dots, \widetilde{\varphi} (\overline{v}_{i-1}), [\overline{v}_i, \overline{w}_2, \dots, \overline{w}_n], \widetilde{\varphi} (\overline{v}_{i+1}), \dots, \widetilde{\varphi} (\overline{v}_n)),
\end{aligned}$$

for all $\overline{v}_i, \overline{w}_j \in V/Z_\varphi(V), 1 \leq i \leq n$ and $2 \leq j \leq n$. Now, we define

$$T : (Z_\varphi(V), V/Z_\varphi(V), r) \longrightarrow V,$$

such that $T(a, \overline{v}) = a + \theta(\overline{v})$, for all

$$a \in Z_\varphi(V), \quad \overline{v} = v + Z_\varphi(V) = k + Z_\varphi(V) \in V/Z_\varphi(V), \quad k \in K.$$

T is well-defined and it is injective, because if

$$T(a_1, \overline{v}_1) = T(a_2, \overline{v}_2),$$

i.e. $a_1 + k_1 = a_2 + k_2$ then $a_1 - a_2 = k_2 - k_1 \in Z_\varphi(V) \cap K = 0$ implies $(a_1, \overline{v}_1) = (a_2, \overline{v}_2)$.
Also, T is a Hom-Lie algebra morphism as

$$\begin{aligned} T[(a_1, \overline{v}_1), \ldots, (a_n, \overline{v}_n)] &= T(r(\overline{v}_1, \ldots, \overline{v}_n), [\overline{v}_1, \ldots, \overline{v}_n]) \\ &= r(\overline{v}_1, \ldots, \overline{v}_n) + \theta([\overline{v}_1, \ldots, \overline{v}_n]) \\ &= [\theta(\overline{v}_1), \ldots, \theta(\overline{v}_1)] \\ &= [a_1 + \theta(\overline{v}_1), \ldots, a_n + \theta(\overline{v}_n)] \\ &= [T(a_1, \overline{v}_1), \ldots, T(a_n, \overline{v}_n)], \end{aligned}$$

for $(a_i, \overline{v}_i) \in R$. Also, the following diagram commutes

$$\begin{array}{ccc} Z_\varphi(V) \oplus V/Z_\varphi(V) & \xrightarrow{T} & V \\ \psi \downarrow & & \downarrow \varphi \\ Z_\varphi(V) \oplus V/Z_\varphi(V) & \xrightarrow{T} & V \end{array}$$

since

$$\begin{aligned} \varphi T(a, \overline{v}) &= \varphi(a + \theta(\overline{v})) = \varphi(a + k), \\ T\psi(a, \overline{v}) &= T(\varphi(a), \widetilde{\varphi}\,(\overline{v})) = \varphi(a) + \theta(\varphi(v) + Z_\varphi(V)) \\ &\qquad = \varphi(a) + \varphi(k) = \varphi(a + k), \end{aligned}$$

for all $a \in Z_\varphi(V), \overline{v} = v + Z_\varphi(V) = k + Z_\varphi(V) \in V/Z_\varphi(V), k \in K$.

The next lemma gives the connection between two isoclinic Hom-stem n-Hom-Lie algebras.

Lemma 8 *Let (V, φ_1) be a Hom-stem n-Hom-Lie algebra in an isoclinism family of n-Hom-Lie algebras $\mathfrak{C}$. Then for any Hom-stem n-Hom-Lie algebra (W, φ_2) of $\mathfrak{C}$, there exists a factor set r over (V, φ_1) such that*

$$W \cong (Z_{\varphi_1}(V), V/Z_{\varphi_1}(V), r).$$

Proof Let (α, β) be an isoclinism pair of n-Hom-Lie algebras (V, φ) and (W, ψ). Then $\beta(Z_{\varphi_1}(V)) = Z_{\varphi_2}(W)$. By Lemma 7, there exists a factor set s such that $W \cong (Z_{\varphi_2}(W), W/Z_{\varphi_2}(W), s)$. Now, we define the following factor set

$$\begin{array}{llll} r: & V/Z_{\varphi_1}(V) \oplus \cdots \oplus V/Z_{\varphi_1}(V) & \longrightarrow & Z_{\varphi_1}(V) \\ & (\overline{v}_1, \ldots, \overline{v}_n) & \longmapsto & \beta^{-1}(s(\alpha(\overline{v}_1), \ldots, \alpha(\overline{v}_n))), \end{array}$$

for all $\overline{v}_i \in V/Z_{\varphi_1}(V), 1 \leq i \leq n$.
Since α and β are isomorphisms, hence $\alpha\, \widetilde{\varphi}_1 = \widetilde{\varphi}_2\, \alpha$. Now, we show that r is

a factor set by the following way

$$\begin{aligned}
&r\Big([\overline{v}_1,\ldots,\overline{v}_n],\widetilde{\varphi}_1\ (\overline{w}_2),\ldots,\widetilde{\varphi}_1\ (\overline{w}_n)\Big)\\
&=\beta^{-1}\Big(s\big(\alpha[\overline{v}_1,\ldots,\overline{v}_n],\alpha(\widetilde{\varphi}_1\ (\overline{w}_2)),\ldots,\alpha(\widetilde{\varphi}_1\ (\overline{w}_n))\big)\Big)\\
&=\beta^{-1}\Big(s([\alpha(\overline{v}_1),\ldots,\alpha(\overline{v}_n)],\widetilde{\varphi}_2\ \alpha(\overline{w}_2),\ldots,\widetilde{\varphi}_2\ \alpha(\overline{w}_n))\Big)\\
&=\beta^{-1}\Big(\sum_{i=1}^{n}s(\widetilde{\varphi}_2\ \alpha(\overline{v}_1),\ldots,\widetilde{\varphi}_2\ \alpha(\overline{v}_{i-1}),[\alpha(\overline{v}_i),\alpha(\overline{w}_2),\ldots,\alpha(\overline{w}_n)],\\
&\qquad\qquad\widetilde{\varphi}_2\ \alpha(\overline{v}_{i+1}),\ldots,\widetilde{\varphi}_2\ \alpha(\overline{v}_n))\Big)\\
&=\sum_{i=1}^{n}\beta^{-1}s(\alpha\ \widetilde{\varphi}_1\ (\overline{v}_1),\ldots,\alpha\ \widetilde{\varphi}_1\ (\overline{v}_{i-1}),\alpha[\overline{v}_i,\overline{w}_2,\ldots,\overline{w}_n],\alpha\ \widetilde{\varphi}_1\ (\overline{v}_{i+1}),\ldots,\alpha\ \widetilde{\varphi}_1\ (\overline{v_n}))\\
&=\sum_{i=1}^{n}r(\widetilde{\varphi}_1\ (\overline{v}_1),\ldots,\widetilde{\varphi}_1\ (\overline{v}_{i-1}),[\overline{v}_i,\overline{w}_2,\ldots,\overline{w}_n],\widetilde{\varphi}_1\ (\overline{v}_{i+1}),\ldots,\widetilde{\varphi}_1\ (\overline{v}_n)),
\end{aligned}$$

for all $\overline{v}_i,\overline{w}_j\in V/Z_\varphi(V), 1\leq i\leq n$ and $2\leq j\leq n$.
Put $R=(Z_{\varphi_1}(V),V/Z_{\varphi_1}(V),r)$ and $S=(Z_{\varphi_2}(W),W/Z_{\varphi_2}(W),s)$. By Lemma 6, (R,ψ_1) and (S,ψ_2) are n-Hom-Lie algebras. We define $\eta:R\longrightarrow S$ given by $\eta(a,\overline{v})=(\beta(a),\alpha(\overline{v}))$. Clearly, η is a well-defined bijection and also,

$$\begin{aligned}
\eta[(a_1,\overline{v}_1),\ldots,(a_n,\overline{v}_n)]&=\eta(r(\overline{v}_1,\ldots,\overline{v}_n),[\overline{v}_1,\ldots,\overline{v}_n])\\
&=\Big(\beta(r(\overline{v}_1,\ldots,\overline{v}_n)),\alpha([\overline{v}_1,\ldots,\overline{v}_n])\Big)\\
&=\Big(s(\alpha(\overline{v}_1),\ldots,\alpha(\overline{v}_n)),[\alpha(\overline{v}_1),\ldots,\alpha(\overline{v}_n)]\Big)\\
&=[(\beta(a_1),\alpha(\overline{v}_1)),\ldots,(\beta(a_n),\alpha(\overline{v}_n))]\\
&=[\eta(a_1,\overline{v}_1),\ldots,\eta(a_n,\overline{v}_n)],
\end{aligned}$$

for all $a\in Z_\varphi(V),\overline{v}=v+Z_\varphi(V)=k+Z_\varphi(V)\in V/Z_\varphi(V),k\in K$. Also, the following diagram is commutative

$$\begin{array}{ccc}
R & \xrightarrow{\ \eta\ } & S\\
{\scriptstyle\psi_1}\downarrow & & \downarrow{\scriptstyle\psi_2}\\
R & \xrightarrow{\ \eta\ } & S
\end{array}$$

by

$$\begin{aligned}
&\eta\psi_1(a,\overline{v})=\eta(\varphi_1(a),\\
&\widetilde{\varphi}_1\ (\overline{v}))=(\beta\varphi_1(a),\alpha\ \widetilde{\varphi}_1\ (\overline{v})),\\
&\psi_2\eta(a,\overline{v})=\psi_2(\beta(a),\alpha(\overline{v}))=(\varphi_2\beta(a),\widetilde{\varphi}_2\ \alpha(\overline{v})),
\end{aligned}$$

for all $a\in Z_\varphi(V),\overline{v}=v+Z_\varphi(V)=k+Z_\varphi(V)\in V/Z_\varphi(V),k\in K$. Since α and β are isomorphisms, $\beta\varphi_1=\varphi_2\beta$ and $\alpha\ \widetilde{\varphi}_1=\widetilde{\varphi}_2\ \alpha$, we conclude that $\eta\psi_1=\psi_2\eta$. So η is our desired isomorphism and $R\cong S$.

Lemma 9 *Let (V, φ) be an n-Hom-Lie algebra, r and s be two multiplicative factor sets over (V, φ). Assume that*

$$R = (Z_\varphi(V), \frac{V}{Z_\varphi(V)}, r), \quad Z_R = \{(a, 0) \in R : a \in Z_\varphi(V)\}$$

and

$$S = (Z_\varphi(V), \frac{V}{Z_\varphi(V)}, s), \quad Z_S = \{(a, 0) \in S : a \in Z_\varphi(V)\}.$$

If η is an isomorphism from R to S satisfying $\eta(Z_R) = Z_S$, then the restrictions of η on $V/Z_\varphi(V)$ and $Z_\varphi(V)$ define the automorphisms $\mu \in Aut(V/Z_\varphi(V))$ and $\nu \in Aut(Z_\varphi(V))$, respectively.

Proof By Lemma 6, (R, ψ) and (S, ψ) are n-Hom-Lie algebras, so we have n-Hom-Lie algebras R/Z_R and S/Z_S and since η is isomorphism and $\eta(Z_R) = Z_S$, thus η induces $\overline{\eta} : (R/Z_R, \psi_1) \longrightarrow (S/Z_S, \psi_2)$ given by

$$(a, \overline{v}) + Z_R \longmapsto \eta(a, \overline{v}) + Z_S,$$

is an isomorphism in which $\psi_1 : R/Z_R \longrightarrow R/Z_R$ and $\psi_2 : S/Z_S \longrightarrow S/Z_S$ are linear maps defined by

$$\psi_1\Big((a, \overline{v}) + Z_R\Big) = \psi(a, \overline{v}) + Z_R \quad \text{and} \quad \psi_2\Big((a, \overline{v}) + Z_S\Big) = \psi(a, \overline{v}) + Z_S,$$

for $(a, \overline{v}) \in R/Z_R$, respectively. Consider σ_1 and σ_2 as two projection maps in the following diagram given by $\sigma_1(\overline{v}) = (0, \overline{v}) + Z_R$ and $\sigma_2(\overline{v}) = (0, \overline{v}) + Z_S$, for $\overline{v} \in V/Z_\varphi(V)$. Now, we define μ such that the following diagram commutes.

$$\begin{array}{ccc} V/Z_\varphi(V) & \xrightarrow{\mu} & V/Z_\varphi(V) \\ \sigma_1 \downarrow & & \downarrow \sigma_2 \\ R/Z(R) & \xrightarrow{\bar{\eta}} & S/Z(S) \end{array}$$

Thus $\eta(0, \overline{v}) + Z_S = (0, \mu(\overline{v})) + Z_S$, for all $\overline{v} \in V/Z_\varphi(V)$. We prove $\mu \widetilde{\varphi} = \widetilde{\varphi} \mu$; For each $v \in V$, $\overline{v} \in V/Z_\varphi(V)$, one can write

$$(0, \mu \widetilde{\varphi}(\overline{v})) + Z_S = \eta(0, \widetilde{\varphi}(\overline{v})) + Z_S = \eta(0, \varphi(v) + Z_\varphi(V)) + Z_S = \eta\psi(0, \overline{v}) + Z_S.$$

On the other hand

$$\begin{aligned} (0, \widetilde{\varphi} \mu(\overline{v})) + Z_S = \psi(0, \mu(\overline{v})) + Z_S &= \psi(\eta(0, \overline{v}) + t) + Z_S \\ &= \psi\eta(0, \overline{v}) + \psi(v) + Z_S = \psi\eta(0, \overline{v}) + Z_S, \end{aligned}$$

where $t \in Z_S$. Since $\eta\psi = \psi\eta$, we have $(0, \mu \widetilde{\varphi} (\overline{v})) + Z_S = (0, \widetilde{\varphi} \mu(\overline{v})) + Z_S$. By the definition, $\sigma_2(\mu \widetilde{\varphi} (\overline{v})) = \sigma_2(\widetilde{\varphi} \mu(\overline{v}))$, and injectivity of σ_2 implies $\mu \widetilde{\varphi} (\overline{v}) = \widetilde{\varphi} \mu(\overline{v})$. Also,

$$\begin{aligned}
(0, \mu([\overline{v}_1, \ldots, \overline{v}_n])) + Z_S &= \eta(0, [\overline{v}_1, \ldots, \overline{v}_n]) + Z_S \\
&= \eta([(0, \overline{v}_1), \ldots, (0, \overline{v}_n)]) + Z_S \\
&= [\eta(0, \overline{v}_1), \ldots, \eta(0, \overline{v}_n)] + Z_S \\
&= [\eta(0, \overline{v}_1) + Z_S, \ldots, \eta(0, \overline{v}_n) + Z_S] \\
&= [(0, \mu(\overline{v}_1)) + Z_S, \ldots, (0, \mu(\overline{v}_n)) + Z_S] \\
&= [(0, \mu(\overline{v}_1)), \ldots, (0, \mu(\overline{v}_n))] + Z_S \\
&= (0, [\mu(\overline{v}_1), \ldots, \mu(\overline{v}_n)]),
\end{aligned}$$

for $\overline{v}_i \in V/Z_\varphi(V), 1 \leq i \leq n$. Hence $\mu([\overline{v}_1, \ldots, \overline{v}_n]) = [\mu(\overline{v}_1), \ldots, \mu(\overline{v}_n)]$ and μ is an automorphism. Now, consider the map $\widetilde{\eta}: Z_R \longrightarrow Z_S$ is defined by $\widetilde{\eta} (a, 0) = \eta(a, 0)$, for all $a \in Z_\varphi(V)$. It is an isomorphism and we define ν such that the following diagram is commutative

$$\begin{array}{ccc}
Z_\varphi(V) & \xrightarrow{\ \nu\ } & Z_\varphi(V) \\
{\scriptstyle \bar{\sigma}_1}\downarrow & & \downarrow{\scriptstyle \bar{\sigma}_2} \\
Z_R & \xrightarrow{\ \tilde{\mu}\ } & Z_S
\end{array}$$

where $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are projection maps and $\eta(a, 0) = (\nu(a), 0)$, for all $a \in Z_\varphi(V)$. Similarly, one can easily check that ν is automorphism.

Lemma 10 *Let (V, φ) be an n-Hom-Lie algebra and $(R, \psi), (S, \psi), Z_R$ and Z_S be as in lemma 9.*

(i) Consider $\eta : R \longrightarrow S$ is a Hom-Lie algebra isomorphism such that $\eta(Z_R) = Z_S$. Let $\mu \in Aut(V/Z_\varphi(V))$ and $\nu \in Aut(Z_\varphi(V))$ be the automorphisms induced by η. Then there exists a linear map $\gamma : V/Z_\varphi(V) \longrightarrow Z_\varphi(V)$ such that

$$\nu(r(\overline{v}_1, \ldots, \overline{v}_n)) + \gamma[\overline{v}_1, \ldots, \overline{v}_n] = s(\mu(\overline{v}_1), \ldots, \mu(\overline{v}_n)).$$

(ii) If $\mu \in Aut(V/Z_\varphi(V))$ and $\nu \in Aut(Z_\varphi(V))$ and $\delta : V/Z_\varphi(V) \longrightarrow Z_\varphi(V)$ is a linear map such that

$$\nu(r(\overline{v}_1, \ldots, \overline{v}_n)) + \delta[\overline{v}_1, \ldots, \overline{v}_n] = s(\mu(\overline{v}_1), \ldots, \mu(\overline{v}_n)), \quad \delta \widetilde{\varphi} = \varphi\delta,$$

then there exists an isomorphism $\eta : R \longrightarrow S$ which is induced by μ and ν satsfying $\eta(Z_R) = Z_S$.

Proof (i) For all $a \in Z_\varphi(V)$ and $\overline{v} \in V/Z_\varphi(V)$ we have $\eta(a, 0) = (\nu(a), 0)$ and $\eta(0, \overline{v}) + Z_S = (0, \mu(\overline{v})) + Z_S$. Hence

$$\eta(0, \overline{v}) - (0, \mu(\overline{v})) \in Z_S \Rightarrow \eta(0, \overline{v}) - (0, \mu(\overline{v})) = (a_{\overline{v}}, 0),$$

for some $a_{\overline{v}} \in Z_\varphi(V)$. Now, define the map $\gamma : V/Z_\varphi(V) \longrightarrow Z_\varphi(V)$ such that $\gamma(\overline{v}) = a_{\overline{v}}$, for all $\overline{v} = v + Z_\varphi(V) \in V/Z_\varphi(V)$. It is a well-defined linear map and we have

$$\begin{aligned}\eta(a, \overline{v}) &= \eta(a, 0) + \eta(0, \overline{v}) \\ &= (\nu(a), 0) + (0, \mu(\overline{v})) + (\gamma(\overline{v}), 0) \\ &= (\nu(a) + \gamma(\overline{v}), \mu(\overline{v})).\end{aligned}$$

Also,

$$\begin{aligned}\eta[(0, \overline{v}_1), \ldots, (0, \overline{v}_n)] &= [\eta(0, \overline{v}_1), \ldots, \eta(0, \overline{v}_n)] \\ &= [(\gamma(\overline{v}_1), \mu(\overline{v}_1)), \ldots, (\gamma(\overline{v}_n), \mu(\overline{v}_n))] \\ &= (s(\mu(\overline{v}_1), \ldots, \mu(\overline{v}_n)), [\mu(\overline{v}_1), \ldots, \mu(\overline{v}_n)]),\end{aligned}$$

and finally

$$\nu(r(\overline{v}_1, \ldots, \overline{v}_n)) + \gamma[\overline{v}_1, \ldots, \overline{v}_n] = s(\mu(\overline{v}_1), \ldots, \mu(\overline{v}_n)).$$

(ii) We only check that the following diagram commutes, in which $\eta : R \longrightarrow S$ is defined by $\eta(a, \overline{v}) = (\nu(a) + \delta(\overline{v}), \mu(\overline{v}))$,

$$\begin{array}{ccc} R & \xrightarrow{\ \eta\ } & S \\ \psi \downarrow & & \downarrow \psi \\ R & \xrightarrow{\ \eta\ } & S \end{array}$$

$$\begin{aligned}\eta\psi(a, \overline{v}) &= \eta(\varphi(a), \widetilde{\varphi}\,(\overline{v})) = (\nu\varphi(a) + \delta\,\widetilde{\varphi}\,(\overline{v}), \mu\,\widetilde{\varphi}\,(\overline{v})), \\ \psi\eta(a, \overline{v}) &= \psi(\nu(a) + \gamma(\overline{v}), \mu(\overline{v})) = (\varphi\nu(a) + \varphi\delta(\overline{v}), \widetilde{\varphi}\,\mu(\overline{v})).\end{aligned}$$

Since, μ and ν are isomorphisms and $\delta\,\widetilde{\varphi} = \varphi\delta$, so $\eta\psi = \psi\eta$.

The following theorem plays a major role which leads us to deduce the main theorems of this section.

Theorem 1 *Let (V, φ_1) and (W, φ_2) be two finite-dimensional Hom-stem n-Hom-Lie algebras and φ_1 be bijection. Then $V \sim W$ if and only if $V \cong W$.*

Proof Suppose that $V \sim W$. By Lemmas 7 and 8,

$$V \cong (Z_{\varphi_1}(V), V/Z_{\varphi_1}(V), r) = R,$$

and also $W \cong (Z_{\varphi_2}(W), W/Z_{\varphi_2}(W), s) = S$. Now, let (α, β) be isoclinism pair between the n-Hom-Lie algebras (R, ψ_1) and (S, ψ_2). Certainly $Z_R = Z(R)$ and $Z_S = Z(S)$. Let the map $\mu \in Aut(V/Z_{\varphi_1}(V))$ is defined by

$$\alpha((0, \overline{v}) + Z_R) = (0, \mu(\overline{v})) + Z_S,$$

for all $\overline{v} \in V/Z_{\varphi_1}(V)$. Also, $\nu \in Aut(Z_{\varphi_1}(V))$ is the map defined by $\beta(a, 0) = (\nu(a), 0)$, for all $a \in Z_{\varphi_1}(V)$. Let us consider the following commutative diagram

$$\begin{array}{ccccc} V/Z_{\varphi_1}(V) \times \cdots \times V/Z_{\varphi_1}(V) & \xrightarrow{\rho} & R/Z_R \times \cdots \times R/Z_R & \xrightarrow{\theta} & R^2 \\ \downarrow{\mu^n} & & \downarrow{\alpha^n} & & \downarrow{\beta} \\ V/Z_{\varphi_1}(V) \times \cdots \times V/Z_{\varphi_1}(V) & \xrightarrow{\sigma} & S/Z_S \times \cdots \times S/Z_S & \xrightarrow{\xi} & S^2 \end{array}$$

in which

$$\begin{aligned} &\rho(\overline{v}_1, \ldots, \overline{v}_n) = ((0, \overline{v}_1) + Z_R, \ldots, (0, \overline{v}_n) + Z_R)), \\ &\sigma(\overline{v}_1, \ldots, \overline{v}_n) = ((0, \overline{v}_1) + Z_S, \ldots, (0, \overline{v}_n) + Z_S)), \\ &\xi((a_1, \overline{v}_1) + Z_S, \ldots, (a_n, \overline{v}_n) + Z_S) = [(a_1, \overline{v}_1), \ldots, (a_{n+1}, \overline{v}_n)] \\ &\qquad = (s(\overline{v}_1, \ldots, \overline{v}_n), [\overline{v}_1, \ldots, \overline{v}_n]), \\ &\theta((a_1, \overline{v}_1) + Z_R, \ldots, (a_n, \overline{v}_n) + Z_R) = [(a_1, \overline{v}_1), \ldots, (a_n, \overline{v}_n)] \\ &\qquad = (r(\overline{v}_1, \ldots, \overline{v}_n), [\overline{v}_1, \ldots, \overline{v}_n]). \end{aligned}$$

We have

$$\begin{aligned} \beta\theta((0, \overline{v}_1) + Z_R, \ldots, (0, \overline{v}_n) + Z_R)) &= \beta(r(\overline{v}_1, \ldots, \overline{v}_n), [\overline{v}_1, \ldots, \overline{v}_n]) \\ &= \beta[(0, \overline{v}_1), \ldots, (0, \overline{v}_n)], \end{aligned}$$

and further

$$\begin{aligned} \xi\alpha^n((0, \overline{v}_1) + Z_R, \ldots, (0, \overline{v}_n) + Z_R)) &= \xi((0, \mu(\overline{v}_1)) + Z_S, \ldots, (0, \mu(\overline{v}_n)) + Z_S)) \\ &= [(0, \mu(\overline{v}_1)), \ldots, (0, \mu(\overline{v}_n))] \\ &= (s(\mu(\overline{v}_1)), \ldots, \mu(\overline{v}_n)), [\mu(\overline{v}_1)), \ldots, \mu(\overline{v}_n]). \end{aligned}$$

Hence we have

$$\beta[(0, \overline{v}_1), \ldots, (0, \overline{v}_n)] = (s(\mu(\overline{v}_1)), \ldots, \mu(\overline{v}_n)), [\mu(\overline{v}_1)), \ldots, \mu(\overline{v}_n]).$$

The map $\delta : V'/Z_{\varphi_1}(V) \longrightarrow Z_{\varphi_1}(V)$ such that

$$\beta(0, [\overline{v}_1, \ldots, \overline{v}_n] = (\delta([\overline{v}_1, \ldots, \overline{v}_n]), t)$$

where $t \in V/Z_{\varphi_1}(V)$ is considered. Thus we get

$$\nu(r(\overline{v}_1, \ldots, \overline{v}_n) + \delta([\overline{v}_1, \ldots, \overline{v}_n])) = s(\mu(\overline{v}_1), \ldots, \mu(\overline{v}_n)).$$

To apply Lemma 10, we may extend δ to $V/Z_{\varphi_1}(V)$ by assuming that it vanishes on the complement of $V'/Z_{\varphi_1}(V)$ in $V/Z_{\varphi_1}(V)$. Now, we need only to show $\delta \widetilde{\varphi_1} = \varphi_1\delta$. For all $\overline{v}_1, \ldots, \overline{v}_n, t \in V/Z_{\varphi_1}(V)$

$$\delta \widetilde{\varphi_1} ([\overline{v}_1, \ldots, \overline{v}_n], t) = \beta(0, \widetilde{\varphi_1} ([\overline{v}_1, \ldots, \overline{v}_n]) = \beta\psi(0, [\overline{v}_1, \ldots, \overline{v}_n]).$$

Further $\widetilde{\varphi_1}$ is onto, i.e. $\widetilde{\varphi_1}(t') = t$ for any $t' \in V/Z_{\varphi_1}(V)$. So

$$(\varphi_1\delta[\overline{v}_1, \ldots, \overline{v}_n], t) = (\varphi_1\delta[\overline{v}_1, \ldots, \overline{v}_n], \widetilde{\varphi}(t')) = \psi(\delta[\overline{v}_1, \ldots, \overline{v}_n], t') \\ = \psi\tau(0, [\overline{v}_1, \ldots, \overline{v}_n]).$$

Hence $\varphi_1\delta(\overline{v}) = \delta\,\widetilde{\varphi_1}(\overline{v})$, for all $(\overline{v}) \in V'$. Consider $V = V' \oplus U$ and define δ to be zero in U. Then $\varphi_1\delta(\overline{u}) = 0$, for all $u \in U$. Since φ_1 is one-to-one, $\varphi_1(u) \in U$, thus

$$\delta\,\widetilde{\varphi_1}(\overline{u}) = \delta(\varphi_1(u) + Z_{\varphi_1}(V)) = 0.$$

Consequently, $\varphi_1\delta = \delta\,\widetilde{\varphi_1}$ and now we can use Lemma 10 to obtain the result.

Theorem 2 *Let $\mathfrak{C}$ be an isoclinism family of finite-dimensional regular n-Hom-Lie algebras. Then any $V \in \mathfrak{C}$ can be expressed as $V = T \oplus A$, where T is a Hom-stem n-Hom-Lie algebra and A is some finite-dimensional abelian n-Hom-Lie algebra.*

Theorem 3 *Let (V, φ_1) and (W, φ_2) be two n-Hom-Lie algebras with same dimension. Then $V \sim W$ if and only if $V \cong W$.*

The following example shows that the above theorem does not valid for two different dimension n-Hom-Lie algebras.

Example 2 Let (V, φ) be an $(n+1)$-dimensional n-Hom-Lie algebra over a field F defined by

$$[e_2, \ldots, e_{n+1}] = e_1, \quad [e_1, e_3, \ldots, e_{n+1}] = e_2,$$

where $\{e_1, \ldots, e_{n+1}\}$ is a basis for V and all other commutator relations are zero. The linear map φ is defined as follows

$$\varphi(e_1) = e_1, \quad \varphi(e_{2i}) = e_{2i+1}, \quad \varphi(e_{2i+1}) = e_{2i}, \quad 1 \le i \le n.$$

Then $V^2 = \langle e_1, e_2 \rangle$ and $Z_\varphi(V) = 0$ and hence $V/Z_\varphi(V) \cong V$.
Now, let (W, ψ) be an $(n+2)$-dimensional n-Hom-Lie algebra with the basis $\{e_1, \ldots, e_{n+2}\}$ and the commutator relations are defined by

$$[e_2, \ldots, e_{n+1}] = e_1, \quad [e_1, e_3, \ldots, e_{n+1}] = e_2,$$

and all other commutator relations are zero. Also, the linear map is given by

$$\psi(e_1) = e_1, \ \psi(e_{n+2}) = e_{n+2}, \ \psi(e_{2i}) = e_{2i+1}, \ \psi(e_{2i+1}) = e_{2i}, \text{ for } 1 \le i \le n/2.$$

Then $W^2 = \langle e_1, e_2 \rangle$ and $Z_\psi(W) = \langle e_{n+2} \rangle$ and so $W/Z_\psi(W) = \langle \overline{e}_1, \ldots, \overline{e}_{n+1} \rangle$, where $\overline{e}_i = e_i + Z_\psi(W)$. We conclude that $V^2 \cong W^2$ and $V/Z_\varphi(V) \cong W/Z_\psi(W)$ and hence $V \sim W$ while $dim(V) \neq dim(W)$.

References

1. F. Ammar, Z. Ejbehi, A. Makhlouf, Cohomology and deformations of Hom-algebras, J. Lie Theory, 21, 813–836 (2011).
2. J. Arnlind, A. Makhlouf, S. Silvestrov, Construction of n-Lie algebras and n-ary Hom-Nambu-Lie Algebras, J. Math. Phys., 52, 123502 (2011).
3. H. Ataguema, A. Makhlouf, S. Silvestrov, Generalization of n-ary nambu algebras and beyond, J. Math. Phys., 50, (2009).
4. P. Batten, E. Stitzinger, On covers of Lie algebras, Comm. Algebra, 24, 4301–4317 (1996).
5. J. M. Casas, X. Garcia-Martinez, Abelian extensions and crossed modules of Hom-Lie algebas, J. Pure Appl. Algebra, 224, 987–1008 (2020).
6. J. M. Casas, X. Garcia-Martinez, On the capability of Hom-Lie algebas, arXiv: 2101.11522v1, (2021).
7. J. M. Casas, M.A.Insua, N. Pacheco, On universal central extensions of Hom-Lie algebras, Hacet. J. Math. Stat., 44, 277–288 (2015).
8. J. M. Casas, E. Khmaladze, N. P. Rego, A non-abelian tensor product of Hom-Lie algebras, Bulletin of the Malaysian mathematical science society, 40, 1035–1054 (2016).
9. Y. Cheng, Y. Su, (Co)homology and universal central extension of Hom-Leibniz algebras, Acta Math. Sin. (Engl. Ser.), 27, 813–830 (2011).
10. G. J. Ellis, A non-abelian tensor product of Lie algebras, Glasgow Math. Journal, 33, 101–120 (1991).
11. K. Erdmann, M. J. Wildon, Intrduction to Lie algebras, (2006).
12. M. Eshrati , M. R. R. Moghaddam , F. Saeedi, Some properties on Isoclinism in n-Lie algebras, Communications in Algebra, 44, 3005–3019 (2016).
13. V. T. Filippov, n-Lie algebras, Sib. Mat. Zh., 26, 126–140 (1987).
14. P. Hall, The classification of prime-power groups, J. Reine Angew. Math, 182, 130–141 (1940).
15. J. T. Hartwig, D. Larsson, S. D. Silvestrov, Deformations of Lie algebras using σ-derivation, Journal of Algebra, 695, 314–361 (2002).
16. Q. Jin, X. Li, Hom-Lie algebra structures on semi-simple Lie algebras, J. Algebra 319, 1398–1408 (2008).
17. N. S. Karamzadeh, S. N. Hosseini, A. R. Salemkar, The exterior product and homology of Hom-Lie algebras, arXiv preprint arXiv:2104.12767, (2021).
18. Y. E. Kharraf, Classification problem of simple Hom-Lie algebras, arXiv preprint arXiv:2101.11518 (2021).
19. K. Moneyhun, Isoclinisms in Lie algebras, Algebras, Groups and Geometries, 11, 9–22 (1994).
20. P. Niroomand, A Note on the Schur multiplier of a nilpotent Lie algebra, Communications in Algebra, 39, 1293–1297 (2011).
21. P. Niroomand, On the tensor square of non-abelian nilpotent finite dimensional Lie algebras, Linear and Multilinear Algebra, 831–836 (2011).
22. R. N. Padhan, N. Nandi, K. C Pati, Some properties of factor set in regular Hom-Lie algebras, arXiv e-prints (2020), arXiv-2005.
23. F. Parvaneh, M. R. R. Moghaddam, Some properties of n-isoclinism in Lie algebras, Italian J. Pure Appl. Math., 28, 165–176 (2011).
24. A. Salemkar, F. Mirzaei, Characterizing n-isoclinism classes of Lie algebras, Communications in Algebra, 38, 3396–3403 (2010).
25. Y. Sheng, Representations of Hom-Lie algebras, Algebra Represent. Theory, 15, 1081–1098 (2012).
26. L. Song, R. Tang, Cohomologies, deformations and extensions of n-Hom-Lie algebras, Journal of Geometry and Physics, 141, 65–78 (2019).
27. D. Yau, Hom-algebras and homology, J. Lie Theorey, 19, 409–421 (2009).
28. D. Yau, The Hom-Yang-Baxter equation, Hom-Lie algebras and quasi-triangular bialgebras, J. Phys. A, 42, 165202 (12pp) (2009).
29. D. Yau, The Hom-Yang-Baxter equation and Hom-Lie algebras, arXiv:0905.1887.