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Some Results on Isologism of Pairs of Groups

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Abstract Let $\mathcal{V}$ be a variety of groups defined by a set V of laws. Then the verbal subgroup and the marginal subgroup of a group G associated with the variety are denoted by $V(G)$ and $V^*(G)$, respectively. Let (N, G) be a pair of groups in which N is a normal subgroup of G. In the paper, we study the lower and upper $\mathcal{V}$-marginal series of the pair (N, G) and prove some properties of isologism of pairs of groups.

Keywords Pair of groups · Variety · Isologism

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1 Introduction and preliminary

Let F be a free group freely generated by a countable set $\{x_1, x_2, \ldots\}$. Let $\mathcal{V}$ be a variety of groups defined by a subset V of F. Then for any group G we assume that the reader is familiar with the notions of the verbal subgroup $V(G)$ and the marginal subgroup $V^*(G)$, associated with the variety of groups. (see [6,7] for more information).

Let (N, G) be a pair of groups in which N is a normal subgroup of G, then we define $[NV^*G]$ to be the subgroup of G generated by the following set

$$\{v(g_1, g_2, \ldots, g_i n, \ldots, g_r)v(g_1, g_2, \ldots, g_r)^{-1} \mid 1 \leq i \leq r, v \in V, g_i \in G, n \in N\}.$$

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We can see that $[NV^*G]$ is the smallest normal subgroup T of G contained in N such that N/T is contained in $V^*(\frac{G}{T})$. Also, we define

$$V^*(N,G) = \{n \in N \mid v(g_1, g_2, \ldots, g_i n, \ldots, g_r) = v(g_1, \ldots, g_r), \\ \forall\, v \in V,\ g_i \in G,\ 1 \leq i \leq r\}.$$

In particular, if $N = G$, then $V(N,G) = V(G)$ and $V^*(N,G) = V^*(G)$ are ordinary verbal and marginal subgroups of G. (see [5,8] for more information).

In 1976, Leedham-Green and McKay [6] introduced the notion of the product of varieties as follows.

Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of groups defined by the set of laws V and W, respectively. The product $\mathcal{U} = \mathcal{V} * \mathcal{W}$ is the variety of all groups G such that $V(G) \subseteq W^*(G)$. Also, the varbal subgroup of the product $\mathcal{U} = \mathcal{V} * \mathcal{W}$ is $U(G) = [V(G)W^*G]$. (see [4] for more information).

The notion of $\mathcal{V} \vee \mathcal{W}$ is the variety whose set of laws are in $V \cap W$ and also, $[\mathcal{V}, \mathcal{W}]$ consists of all groups whose V-subgroups centralize W-subgroups. Moreover, $\mathcal{V}\mathcal{W}$ is the variety of groups such that are extensions of a group in $\mathcal{V}$ by a group in $\mathcal{W}$.

Let (N,G) and (M,H) be pairs of groups. An homomorphism from (N,G) to (M,H) is a homomorphism $f : G \to H$ such that $f(N) \subseteq M$. We say that (N,G) and (M,H) are isomorphic and write $(N,G) \simeq (M,H)$, if f is an isomorphism and $f(N) = M$. Let (N,G) and (M,H) be two pairs of groups and $\mathcal{V}$ be a variety of groups defined by the set of laws V. An $\mathcal{V}$-isologism between (N,G) and (M,H) is a pair of isomorphism (α, β) with $\alpha : G/V^*(N,G) \to H/V^*(M,H)$ and $\beta : V(N,G) \to V(M,H)$, such that

$$\alpha\big(N/V^*(N,G)\big) = M/V^*(M,H),$$

and for every $v \in V$, $n \in N$ and $g_1, \ldots, g_r \in G$

$$\beta\big(v(g_1, \cdots, g_i n, \cdots, g_r)v(g_1, \cdots, g_r)^{-1}\big) = \\ v(h_1, \cdots, h_i m, \cdots, h_r)v(h_1, \cdots, h_r)^{-1},$$

whenever, $h_i \in \alpha\big(g_i V^*(N,G)\big)$ and $m \in \alpha\big(nV^*(N,G)\big)$. We say that (N,G) and (M,H) are $\mathcal{V}$-isologic, if there exists an $\mathcal{V}$-isologism between them. In this case we write $(N,G) \sim_{\mathcal{V}} (M,H)$.

If $\mathcal{V}$ is the variety of abelian groups or nilpotent groups of class at most n, then $\mathcal{V}$-isologism coincides with isoclinism and n-isoclinism between pairs of groups. In addition, if $N = G$ and $M = H$, then $\mathcal{V}$-isologism between two pairs of groups is an $\mathcal{V}$-isologism between G and H. (see [1–3] for more information).

2 The main results

In this section, we generalize some properties of isologism of groups to a pair of groups. First of all, we discuss some preliminaries which are needed for the proof of our results. The following lemma is similar to Lemma 1 of [2].

Lemma 1 *If (N,G) is a pair of groups and $M \trianglelefteq G$ such that $M \leqslant N$, then*

(a) $V(V^*(N,G)) = \langle e\rangle$ *and* $V^*\left(\dfrac{N}{V(N,G)}, \dfrac{G}{V(N,G)}\right) = \dfrac{N}{V(N,G)}$,

(b) $V(N,G) = \langle e\rangle$ *if and only if* $V^*(N,G) = N$ *if and only if* $G \in \mathcal{V}$,

(c) $[KV^*G] = \langle e\rangle$ *if and only if* $K \subseteq V^*(N,G)$,

(d) $V\left(\dfrac{N}{K}, \dfrac{G}{K}\right) = \dfrac{V(N,G)K}{K}$ *and* $V^*\left(\dfrac{N}{K}, \dfrac{G}{K}\right) \supseteq \dfrac{V^*(N,G)K}{K}$,

(e) $V(K) \subseteq [KV^*G] \subseteq K \cap V(N,G)$,

(f) If $K \cap V(N,G) = \langle e\rangle$, *then* $K \subseteq V^*(N,G)$ *and* $V^*\left(\dfrac{N}{K}, \dfrac{G}{K}\right) = \dfrac{V^*(N,G)}{K}$,

(g) If $[K,G] \subseteq V^*(N,G)$, *then* $[V(N,G),K] = \langle e\rangle$. *In particular*

$$[V(N,G), V^*(N,G)] = \langle e\rangle.$$

Theorem 1 *([2], Theorem 2) Let (N_1,G_1) and (N_2,G_2) be pairs of groups. Then $(N_1,G_1) \sim_{\mathcal{V}} (N_2,G_2)$ if and only if there exists a pair (N,G) of groups and there exists normal subgroups M_1 and M_2 of G with $M_1 \subseteq N$ and $M_2 \subseteq N$ such that* $(N_1,G_1) \simeq \left(\dfrac{N}{M_1}, \dfrac{G}{M_1}\right)$, $(N_2,G_2) \simeq \left(\dfrac{N}{M_2}, \dfrac{G}{M_2}\right)$, *and*

$$(N_1,G_1) \sim_{\mathcal{V}} (N,G) \sim_{\mathcal{V}} (N_2,G_2).$$

Lemma 2 *([2] , Lemma 5) Let (N,G) be a pair of groups. If M is a normal subgroup of G with $M \leq N$ and H is a subgroup of G, then*

(a) $(H\cap N, H) \sim_{\mathcal{V}} \big((H\cap N)V^*(N,G), HV^*(N,G)\big)$. *In particular if*

$$G = HV^*(N,G),$$

then $(H \cap N, H) \sim_{\mathcal{V}} (N,G)$. Conversely, if $\frac{H}{V^*(H\cap N,H)}$ satisfies the ascending chain condition on normal subgroups and $(H \cap N, H) \sim_{\mathcal{V}} (N,G)$, then $G = HV^*(N,G)$.

(b) $(N/M, G/M) \sim_{\mathcal{V}} \big(N/M \cap V(N,G), G/M \cap V(N,G)\big)$. *In particular if*

$$M \cap V(N,G) = \langle e\rangle,$$

then $(N,G) \sim_{\mathcal{V}} (\frac{N}{M}, \frac{G}{M})$. Conversely, if $V(N,G)$ satisfies the ascending chain condition on normal subgroups and $(N,G) \sim_{\mathcal{V}} (\frac{N}{M}, \frac{G}{M})$, then

$$M \cap V(N,G) = \langle e\rangle.$$

Definition 1 Let (N,G) be a pair of groups, $\mathcal{V}$ and $\mathcal{W}$ are two varieties of groups defined by the sets of laws V and W, respectively, then the product $\mathcal{V} * \mathcal{W}$ is the variety of all groups G such that $V(N,G) \subseteq W^*(N,G)$.

Lemma 3 *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of groups and put $\mathcal{U} = \mathcal{V} * \mathcal{W}$. Then the following are equivalent.*

(a) For any pair of groups (N, G):

$$\frac{U^*(N,G)}{W^*(N,G)} \subseteq V^*\left(\frac{N}{W^*(N,G)}, \frac{G}{W^*(N,G)}\right),$$

(b) For any pair of groups (N, G) and $K \trianglelefteq G$:

$$[[KV^*G]W^*G] \subseteq [KU^*G].$$

Moreover, the equality sign holds in (a) if and only if the equality sign holds in (b).

Proof $(a) \Rightarrow (b)$: *Let* $\overline{K} = \dfrac{K}{[KU^*G]}$. *So,* $\overline{K} \subseteq U^*\left(\dfrac{N}{K}, \dfrac{G}{K}\right)$. *We can see that*

$$[\overline{K}W^*\left(\frac{N}{K}, \frac{G}{K}\right)V^*\frac{G}{K}] \subseteq W^*\left(\frac{N}{K}, \frac{G}{K}\right).$$

Hence, by using Lemma 1(c) we have $[[\overline{K}V^*\overline{G}]W^*\overline{G}] = \langle \overline{e} \rangle$ *and so,*

$$[[KV^*G]W^*G] \subseteq [KU^*G].$$

$(b) \Rightarrow (a)$: *Now, put* $K = U^*(N, G)$. *By using Lemma 1(c) we have*

$$\frac{U^*(N,G)}{W^*(N,G)} \subseteq V^*\left(\frac{N}{W^*(N,G)}, \frac{G}{W^*(N,G)}\right).$$

Now, assume that for any pair of group (N, G), *we have*

$$\frac{U^*(N,G)}{W^*(N,G)} = V^*\left(\frac{N}{W^*(N,G)}, \frac{G}{W^*(N,G)}\right).$$

Suppose that $K \trianglelefteq G$. *By the first part of the proof* $[[KV^*G]W^*G] \subseteq [KU^*G]$. *Put* $\overline{K} = \dfrac{K}{[[KV^*G]W^*G]}$. *Then*

$$\frac{\overline{K}W^*(N,G)}{W^*(N,G)} \subseteq V^*\left(\frac{\frac{N}{K}}{W^*(\frac{N}{K}, \frac{G}{K})}, \frac{\frac{G}{K}}{W^*(\frac{N}{K}, \frac{G}{K})}\right).$$

Now, $[\overline{K}U^*\dfrac{G}{N}] = \langle \overline{e} \rangle$. *So,* $[KU^*G] \subseteq [[NV^*G]W^*G]$. *Finally, assume that*

$$[[KV^*G]W^*G] = [KU^*G].$$

Put

$$V^*\left(\frac{N}{W^*(N,G)}, \frac{G}{W^*(N,G)}\right) = \frac{M}{W^*(N,G)}.$$

Then $M \trianglelefteq G$ *and* $[MV^*G] \subseteq W^*(N,G)$. *Hence, by using lemma 1(c)*

$$[[MV^*G]W^*G] = \langle e \rangle.$$

On the other hand, $[[MV^*G]W^*G] = [MU^*G]$. *Thus,* $M \subseteq U^*(N,G)$. *Therefore,*

$$V^*\left(\frac{N}{W^*(N,G)}, \frac{G}{W^*(N,G)}\right) \subseteq \frac{U^*(N,G)}{W^*(N,G)},$$

and so, the equality sign holds in (b).

Theorem 2 *Let* (N,G) *be a pair of groups,* $\mathcal{V}, \mathcal{W}$ *are two varieties of groups defined by the sets of laws* V *and* W *and put* $\mathcal{U} = \mathcal{V} * \mathcal{W}$. *Then for any pair of groups* (N,G) *the following hold.*

(a) $W^*(N,G) \subseteq U^*(N,G)$.

(b) $\dfrac{U^*(N,G)}{W^*(N,G)} \subseteq V^*\left(\dfrac{N}{W^*(N,G)}, \dfrac{G}{W^*(N,G)}\right) \subseteq U^*\left(\dfrac{N}{W^*(N,G)}, \dfrac{G}{W^*(N,G)}\right)$.

Proof If (N,G) *is a pair of groups such that* $G \in \mathcal{W}$, *then* $N = W^*(N,G)$. *Also,* $V(N,G) \subseteq W^*(N,G) = N$. *Thus,* $G \in \mathcal{U}$. *Hence,* $\mathcal{W} \subseteq \mathcal{U}$. *It follows that* $W^*(N,G) \subseteq U^*(N,G)$, *which proves* (a). *Now, if* $G \in \mathcal{V}$, *then* $V(N,G) = \langle e \rangle$. *So,* $U(N,G) = [V(N,G)W^*G] = \langle e \rangle$. *Thus,* $G \in \mathcal{U}$ *and so,* $\mathcal{V} \subseteq \mathcal{U}$. *Now, let* $K \trianglelefteq G$, *if* $v(g_1, g_2, \ldots, g_i n, \ldots, g_r)v(g_1, g_2, \ldots, g_r)^{-1}$ *and* $w(g_1, g_2, \ldots, g_i n, \ldots, g_s)v(g_1, g_2, \ldots, g_s)^{-1}$ *are words in* $V(N,G)$ *and* $W(N,G)$, *respectively, then, the laws which determine* $\mathcal{U}$ *are given by*

$$w(g_1, g_2, \ldots, g_i v(g_{s+1}, \ldots, g_{s+r}), \ldots, g_j n, \ldots, g_s)w(g_1, \ldots, g_s)^{-1}$$

where, $1 \leq i \leq s$, $g_i \in G$ *and* $n \in N$. *A generating element of* $[[KV^*G]W^*G]$ *is of the following form*

$$w(g_1, \ldots, g_i v(g_{s+1}, \ldots, g_{s+j}k, \ldots, g_{s+r})v(g_{s+1}, \ldots, g_{s+r})^{-1}, \ldots, g_s)w(g_1, \ldots, g_s)^{-1}, \tag{1}$$

where, $g_1, \ldots, g_{s+r} \in G$, $k \in K$, $1 \leq i \leq s$ *and* $1 \leq j \leq r$. *Put*

$$v = v(g_{s+1}, \ldots, g_{s+r}),$$

and $v' = v(g_{s+1}, \ldots, g_{s+j}k, \ldots, g_{s+r})$. *Then the element in* (1) *takes form*

$$\begin{aligned}
&w(g_1, \ldots, g_i v' v^{-1}, g_{i+1}, \ldots, g_s)w(g_1, \ldots, g_s)^{-1} = \\
&w(g_1, \ldots, g_i v^{-1} v v' v^{-1}, g_{i+1}, \ldots, g_s) \\
&w(g_1, \ldots, g_i v^{-1}, g_{i+1}, \ldots, g_s)^{-1}.w(g_1, \ldots g_i v^{-1}, g_{i+1}, \ldots, g_s)w(g_1, \ldots, g_s)^{-1}
\end{aligned}$$

and this is an element of $[KU^*G]$. *Thus,* $[[KV^*G]W^*G] \subseteq [KU^*G]$.

Corollary 1 *Let* $\mathcal{V}$ *and* $\mathcal{W}$ *be varieties of groups and put* $\mathcal{U} = \mathcal{V} * \mathcal{W}$. *Let* $K \trianglelefteq G$. *Then, the following hold.*

(a) If $K \subseteq U^*(N,G)$, *then* $[KV^*G] \subseteq W^*(N,G)$.

(b) If $K \cap V^*(N,G) = \langle e \rangle$, *then* $K \cap U^*(N,G) \subseteq V^*(N,G)$.

Proof (*a*) *If* $K \subseteq U^*(N,G)$, *then by Lemma 1(c),* $[KU^*G] = \langle e \rangle$. *By lemma 3 and Theorem 2(b), we get* $[[KV^*G]W^*G] = \langle e \rangle$. *Again by Lemma 1(c),* $[KV^*G] \subseteq W^*(N,G)$.
(*b*) *We can see that*

$$[(K \cap U^*(N,G))V^*G] \subseteq K \cap [U^*(N,G)V^*G] \subseteq K \cap W^*(N,G).$$

Thus, if $K \cap W^*(N,G) = \langle e \rangle$, *then by Lemma 1(c) the proof is completed.*

Corollary 2 *Let* $\mathcal{V}, \mathcal{W}$ *and* $\mathcal{U}$ *be any varieties of groups. Then*

$$\mathcal{V} * (\mathcal{W} * \mathcal{U}) \subseteq (\mathcal{U} * \mathcal{W}) * \mathcal{U}.$$

Proof Let $G \in \mathcal{V} * (\mathcal{W} * \mathcal{U})$. *Put* $T = \mathcal{W} * \mathcal{U}$. *Thus,* $V(N,G) \subseteq T^*(N,G)$. *Now,* $[V(N,G)W^*G] \subseteq U^*(N,G)$. *Thus,* $S(N,G) \subseteq U^*(N,G)$, *where* $\mathcal{S} = \mathcal{V} * \mathcal{W}$, *as required.*

Corollary 3 *Let* $\mathcal{V} \subseteq \mathcal{V}_1$ *and* $\mathcal{W} \subseteq \mathcal{W}_1$ *be varieties of groups. Then the following hold.*

(a) $\mathcal{V} * \mathcal{W} \subseteq \mathcal{V}_1 * \mathcal{W}_1$.
(b) $\mathcal{V} * \mathcal{A} \supseteq \mathcal{A} * \mathcal{V}$, *where* $\mathcal{A}$ *is the variety of all abelian groups.*
(c) For any $m, n \geq 0$, $\mathcal{V} * \eta_{m+n} = (\mathcal{V} * \eta_m) * \eta_n$, *where* η_c *is the variety of all nilpotent groups of class at most c.*

Proof (a) If $G \in \mathcal{V} * \mathcal{W}$, *then* $V_1(N,G) \subseteq V(N,G) \subseteq W^*(N,G) \subseteq W_1(N,G)$. *Thus,* $G \in \mathcal{V}_1 * \mathcal{W}_1$.
(b) Let $G \in \mathcal{A} * \mathcal{V}$. *Thus,* $[N,G] \subseteq V^*(N,G)$. *Then* $V(N,G) \subseteq A(N,G)$ *and so,* $G \in \mathcal{V} * \mathcal{A}$.
(c) It is follows from the fact that for any groups G and $m, n \geq 0$,

$$\frac{Z_{m+n}(N,G)}{Z_n(N,G)} = Z_m\left(\frac{N}{Z_n(N,G)}\right).$$

Proposition 1 *Let* $\mathcal{V}$, $\mathcal{W}$ *and* $\mathcal{U}$ *be varieties of groups. Then the following hold.*

(a) $\mathcal{V} \vee \mathcal{W} \subseteq \mathcal{V} * \mathcal{W} \subseteq \mathcal{V}\mathcal{W}$.
(b) If $\mathcal{V} \subseteq \mathcal{U} * \mathcal{A}$, *then* $\mathcal{V} * \mathcal{W} \subseteq [\mathcal{U}, \mathcal{W}]$. *In particular* $\mathcal{V} * \mathcal{W} \subseteq [\mathcal{V}, \mathcal{W}]$.

Proof (a) By Lemma 1(e), we have $[V(N,G)W^*G] \subseteq V(N,G) \cap W(N,G)$ *for any pair* (N,G) *of groups. Hence,* $\mathcal{V} \vee \mathcal{W} \subseteq \mathcal{V} * \mathcal{W}$. *Assume that* $G \in \mathcal{V} * \mathcal{W}$. *Thus,* $V(N,G) \subseteq W^*(N,G)$. *By Lemma 1(a), we get* $W(V(N,G)) = \langle e \rangle$. *So,* $G \in \mathcal{W}\mathcal{V}$. *we conclude that* $\mathcal{V} * \mathcal{W} \subseteq \mathcal{V}\mathcal{W}$.
(b) Let $G \in \mathcal{V} * \mathcal{W}$. *So,* $V(N,G) \subseteq W^*(N,G)$. *By hypothesis*

$$[U(N,G), N] \subseteq V(N,G),$$

and so, $[U(N,G), N] \subseteq W^*(N,G)$ *and by Lemma 1(g),*

$$[W(N,G), U(N,G)] = \langle e \rangle.$$

Thus, $G \in [\mathcal{U}, \mathcal{W}]$. *This implies that* $\mathcal{V} * \mathcal{W} \subseteq [\mathcal{U}, \mathcal{W}]$ *and so,* $\mathcal{V} \subseteq \mathcal{V} * \mathcal{A}$. *By setting* $\mathcal{U} = \mathcal{V}$, *the result is held.*

Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of groups and put $\mathcal{U} = \mathcal{V} * \mathcal{W}$. For a pair (N, G) of groups, let

$$\Delta_{\mathcal{V},\mathcal{W}}(N,G) = \frac{V^*\left(\dfrac{N}{W^*(N,G)}, \dfrac{G}{W^*(N,G)}\right)}{\dfrac{U^*(N,G)}{W^*(N,G)}}.$$

In other words, $\Delta_{\mathcal{V},\mathcal{W}}(N,G)$, measures to what extend the group

$$V^*\left(\frac{N}{W^*(N,G)}, \frac{G}{W^*(N,G)}\right),$$

deviates from the group $\dfrac{U^*(N,G)}{W^*(N,G)}$, following Lemma 3.

Theorem 3 *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of groups and put $\mathcal{U} = \mathcal{V} * \mathcal{W}$. Assume that $(N, G) \sim_{\mathcal{U}} (H_1, H_2)$. Then $\Delta_{\mathcal{V},\mathcal{W}}(N,G) \simeq \Delta_{\mathcal{V},\mathcal{W}}(K,H)$.*

Proof By Theorem 1, we may assume that

$$(H_1, H_2) = \left(\frac{N}{M}, \frac{G}{M}\right),$$

for some normal subgroup M of G such that $M \leqslant N$ and $M \cap U(N,G) = \langle e \rangle$. By Lemma 1(f), we have $K \subseteq U^*(N,G)$ and

$$U^*\left(\frac{N}{M}, \frac{G}{M}\right) = \frac{U^*(N,G)}{M}.$$

Put $W^*\left(\dfrac{N}{M}, \dfrac{G}{M}\right) = \dfrac{K}{M}$ such that $K \trianglelefteq G$ and $MW^*(N,G) \subseteq K$. Thus,

$$\Delta_{\mathcal{V},\mathcal{W}}(N/M, G/M) = \frac{V^*\left((N/M)/W^*\left(N/M, G/M\right), (G/M)/W^*\left(N/M, G/M\right)\right)}{U^*\left(N/M, G/M\right)/W^*\left(N/M, G/M\right)}$$

$$\cong \frac{V^*\left(\dfrac{N}{W^*(N,G)}\Big/\dfrac{K}{W^*(N,G)}, \dfrac{G}{W^*(N,G)}\Big/\dfrac{K}{W^*(N,G)}\right)}{\dfrac{U^*(N,G)/W^*(N,G)}{K/W^*(N,G)}}. \tag{2}$$

We have $[KW^*G] \subseteq M$ and so,

$$[(K \cap V(N,G))W^*G] \subseteq [KW^*G] \cap [V(N,G)W^*G] \subseteq M \cap U(N,G) = \langle e \rangle.$$

Therefore, Lemma 1(c) gives $K \cap V(N,G) \subseteq W^*(N,G)$, where

$$\frac{K}{W^*(N,G)} \cap V\left(\frac{N}{W^*(N,G)}, \frac{G}{W^*(N,G)}\right) = \langle \bar{e} \rangle.$$

An application of Lemma 1(f), again shows that

$$V^*\left(\frac{N}{W^*(N,G)}\Big/\frac{K}{W^*(N,G)},\frac{G}{W^*(N,G)}\Big/\frac{K}{W^*(N,G)}\right)= \frac{V^*\Big(N/W^*(N,G),G/W^*(N,G)\Big)}{K/V^*(N,G)}.$$

Now, by (2)*, we obtain* $\Delta_{\mathcal{V},\mathcal{W}}(N,G)\cong\Delta_{\mathcal{V},\mathcal{W}}(N/M,G/M)$.

Theorem 4 *Let* $\mathcal{V}$ *and* $\mathcal{W}$ *be varieties of groups. Put* $\mathcal{U}=\mathcal{V}*\mathcal{W}$. *Suppose that* $(N,G)\sim_{\mathcal{U}}(K,H)$. *Then the following hold.*

(a) $(N/V^*(N,G),G/V^*(N,G))\sim_{\mathcal{V}}(K/V^*(K,H),H/V^*(K,H))$.
(b) $V(N,G)\sim_{\mathcal{W}}V(K,H)$.

Proof (a) Put $K/M=V^*(N/M,G/M)$, *so that* $K\trianglelefteq G$ *and*

$$MW^*(N,G)\subseteq K.$$

Now, we show that $K\cap V(N,G)\subseteq W^*(N,G)$. *Indeed it implies that*

$$(K\cap V(N,G))W^*(N,G)=W^*(N,G),$$

whence

$$K/W^*(N,G)\cap V(N/W^*(N,G),G/W^*(N,G))=\langle\bar{e}\rangle,$$

by Lemma 1(d). Thus, by Lemma 2(b) we have

$$\begin{aligned}(N/W^*(N,G),G/W^*(N,G))&\sim_{\mathcal{V}}(N/W^*(N,G),G/W^*(N,G))/(K/W^*(N,G))\\&\cong N/K\\&\cong(N/M)/(K/M)\\&\cong(N/M)/W^*(N/M,G/M).\end{aligned}$$

which is precisely what we want to prove. certainly

$$[(K\cap V(N,G))W^*G]\subseteq[KW^*G]\cap[V(N,G)W^*G]\subseteq M\cap U(N,G)=\langle e\rangle.$$

So, indeed by Lemma 1(c), we have $K\cap V(N,G)\subseteq W^*(N,G)$.

(b) We show that

$$V(N,G)\sim_{\mathcal{W}}V(N/M,G/M)=V(N,G)K/K\cong V(N,G)/(M\cap V(N,G)).$$

Now, we have $M\cap V(N,G)\cap W(V(N,G))=M\cap W(V(N,G))$. *So, by Proposition 1(a), we get* $U(N,G)\supseteq W(V(N,G))$. *Since,*

$$M\cap U(N,G)=\langle e\rangle,$$

we obtain $M\cap W(V(N,G))=\langle e\rangle$.

Corollary 4 *Let* $n\geq 0$ *and* $(N,G)\underset{n}{\sim}(K,H)$. *Then for each* $i\in\{0,\ldots,n\}$, *the following hold*

(a) $(N/Z_i(N,G), G/Z_i(N,G)) \underset{n-i}{\sim} (K/Z_i(K,H), H/Z_i(K,H))$.
(b) $[N, {}_iG] \underset{n-i}{\sim} [K, {}_iH]$.

Proof *By Corollary 3(c), we have* $\eta_i * \eta_{n-i} = \eta_n = \eta_{n-i} * \eta_i$ *for any* i *with* $0 \leq i \leq n$. *Thus, the result follows from Theorem 4.*

Lemma 4 *Let* $\mathcal{V}$ *and* $\mathcal{W}$ *be varieties of group. Then the following are equivalent*

(a) $\mathcal{V} \subseteq \mathcal{W}$.
(b) For any two pairs of groups (N,G) *and* (K,H), $(N,G) \underset{\mathcal{V}}{\sim} (K,H)$ *implies* $(N,G) \underset{\mathcal{W}}{\sim} (K,H)$.

Proof $(a) \Rightarrow (b)$: *Let* $(N,G) \underset{\mathcal{V}}{\sim} (K,H)$. *We may assume by Theorem 1 that* $(K,H) \cong (N/M, G/M)$ *for some* $M \trianglelefteq G$ *with* $M \cap V(N,G) = e$. *As* $\mathcal{V} \subseteq \mathcal{W}$, *we have* $W(N,G) \subseteq V(N,G)$, *whence* $M \cap W(N,G) = e$. *By Lemma 2(b), we get* $(N,G) \underset{\mathcal{W}}{\sim} (N/M, G/M)$.

$(b) \Rightarrow (a)$: *Let* $G \in \mathcal{V}$, *so* $(N,G) \underset{\mathcal{V}}{\sim} (e,e)$. *By hypothesis this implies* $(N,G) \underset{\mathcal{W}}{\sim} (e,e)$. *Thus, in particular* $W(N,G) = e$. *Again Lemma 1(b) shows that* $G \in \mathcal{W}$.

Let χ denote a class of groups which is invariant under 1-isoclinism.

Corollary 5 *Let* $\mathcal{W}$ *be a variety and* $\mathcal{U}$ *a subvariety of* $\mathcal{A} * \mathcal{W}$. *Suppose that* $(N,G) \underset{\mathcal{U}}{\sim} (K,H)$. *Then the following hold.*

(a) $N/V^*(N,G) \in \chi$ *if and only if* $K/V^*(K,H) \in \chi$.
(b) $W(N,G) \in \chi$ *if and only if* $W(K,H) \in \chi$.

Proof *By Corollary 3(b)* $\mathcal{U} \subseteq \mathcal{A} * \mathcal{W} \subseteq \mathcal{W} * \mathcal{A}$. *Hence, by Theorem 4 and Lemma 4, the proof is completed.*

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