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Computing the Spectrum of $L^t(G)$ for A Regular Graph

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Abstract If $L(G)$ is the line graph of G, it is difficult to get the adjacency matrix of $L^t(G) = L(L(L \ldots L(G))); t \geq 3$ and also its spectrum. In this paper, we present a formula to compute the spectrum of $L^t(G)$, for each positive integer t, where G is a regular graph.

Keywords Line graph · Simple graph · Adjacency matrix · Eigenvalue of a matrix · Characteristic polynomial, Spectra of a graph

Mathematics Subject Classification (2010) 05C76 · 05E30

1 Introduction

Modeling problems in various sciences can somehow lead to the study of corresponding graphs [1,5,10]. Suppose $G(V,E)$ is a graph with n vertices $\{v_1, v_2, \ldots, v_n\}$ and m edges $\{e_1, e_2, \ldots, e_m\}$. All over this paper a graph is undirected, without loops and multiple edges, unless indicated otherwise. When v_i and v_j are the endpoints of an edge, they are adjacent. Such vertices are also called neighbors of each other. We say the graph is complete if any two vertices are adjacent. The complement $\bar{G}$ of a graph G is the graph with same vertex set, but with complementary edge set, that is, two vertices are adjacent in $\bar{G}$ if they are not adjacent in G. The degree of a vertex is its number of neighbors. If all vertices have the same degree then the graph is called regular.

The adjacency matrix A of a graph $G, A(G)$, is a square matrix of order n whose rows and columns correspond to the vertices of G such that $(A)_{ij} = 1$ if and only if v_i and v_j are adjacent and $(A)_{ij} = 0$, otherwise. The incidence

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matrix of G is the $0-1$ matrix M, with rows indexed by the vertices and columns indexed by the edges, where $(M)_{ij}=1$ when vertex v_i is an endpoint of edge e_j. The line graph $L(G)$ of a graph G is constructed by taking the edges of G as vertices of $L(G)$, and joining two vertices in $L(G)$ whenever the corresponding edges in G have a common vertex [5]. The eigenvalues of a matrix A are the numbers λ such that $Ax=\lambda x$ has a nonzero solution vector; each such solution is an eigenvector associated with A. The eigenvalues of a graph are the eigenvalues of its adjacency matrix A [2,6,11]. These are the roots $\lambda_1,\lambda_2,\ldots,\lambda_n$ of the characteristic polynomial

$$\chi(G;\lambda)=\det(\lambda I-A)=\prod_{i=1}^{n}(\lambda-\lambda_i)\,.$$

The spectrum is the list of distinct eigenvalues with their multiplicities $b_1,b_2,\ldots,b_t$, we write

$$\mathrm{Spec}(G)=\begin{pmatrix}\lambda_1 & \lambda_2 & \cdots & \lambda_t\\ b_1 & b_2 & \cdots & b_t\end{pmatrix}.$$

According to the above definitions, if G has adjacency matrix A, then $\bar{G}$ has adjacency matrix $\bar{A}=J-I-A$ in which J denote the matrix each of whose entries is $+1$ and I denote the identity matrix. If G is k-regular with eigenvalues $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_n$, then the eigenvalues of the complement are $n-k-1,-1-\lambda_2,\ldots,-1-\lambda_n$. It is difficult to draw $L^t(G)=L(L(L\ldots L(G)));t\geq 3$, for graphs and to get its adjacency matrix and also its spectrum [3,4]. In this paper, we will describe a formula to obtain the spectrum of $L^t(G)$, for each positive integer t, where G is a regular graph. Finally, we determine the spectrum of the complete graph with 4 vertices using the given formula. However, this graph is the smallest regular graph (note that a cycle is isomorphic to its line graph [5,9]), it is difficult to draw $L^t(G);t\geq 3$, for this graph and to get its spectrum. As noted, the line graph $L(G)$ has a vertex for each edge of G, and $e_i,e_j\in E(G)$ are neighbor in $L(G)$ if they have a common endpoint in G. If G be a simple graph, the edges of $L(G)$ correspond to the incident pairs of edges in G. Such pairs share exactly one vertex, and each vertex $v_i\in V(G)$ contributes exactly $\binom{d_i}{2}$ such incident pairs where d_i is the degree of v_i. Therefore, for a simple graph G, we have

$$|E(L(G))|=\sum_{i=1}^{n}\binom{d_i}{2}.$$

We note that if G be a loopless graph with $a_{ij}+1$ edges between v_i and $v_j, i,j=1,\ldots,n$, the number of edges in $L(G)$ is

$$\begin{aligned}|E(L(G))| = &\sum_{v_i\in V(G)} \binom{d_i^\circ}{2} \\ &+\sum_{i=1}^{n}\left(\sum_{j\neq i,j=1}^{n} a_{ij}\right) d_i \\ &-\frac{3}{2}\left(\sum_{i=1}^{n}\left(\sum_{j\neq i,j=1}^{n} a_{ij}\left(a_{ij}+1\right)\right)\right. \\ &-\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} a_{ij}a_{jk}\right), \qquad (1)\end{aligned}$$

where d_i° is the degree of vertex v_i in the simple graph G° formed by deleting a_{ij} edges between v_i and $v_j, i,j=1,\ldots,n$, in G. In the above formula, the coefficient of d_i equals to the sum of the number of the additional edges in which v_i is their endpoint. Furthermore, the last term on the right of formula is the product of pairwise additional edges in each vertex.

2 The spectra of line graphs

The spectra of line graphs have been studied extensively. We will describe a formula to obtain the spectrum of $L^t(G) = L(L(L\ldots L(G)\ldots))$, for each positive integer t, where G is a regular graph. First, we represent outline the basic results in this field. Let G be a k-regular graph with n vertices and m edges. Since a given edge e in G is adjacent with $k-1$ edges in each of its endpoint, so the line graph is $(2k-2)$-regular. So, briefly, we have

$$\begin{aligned} G: |V| &= n \\ |E| &= m \\ k- &\text{ regular} \end{aligned}$$

and

$$\begin{aligned} L(G): n^{(1)} &= m = \frac{1}{2}nk \\ m^{(1)} &= \sum_{i=1}^{n}\binom{d_i}{2} = \frac{1}{2}nk(k-1) \\ k^{(1)} &= 2(k-1), \end{aligned}$$

where $n^{(1)}, m^{(1)}$ and $k^{(1)}$ denote the number of vertices, edges and regularity of the line graph $L(G)$, respectively.

Theorem 1 *Let G be a k-regular graph, then [1]*
(1) k is an eigenvalue of G.
(2) If G is connected, then the multiplicity of k is one.
(3) For any eigenvalue λ of G, we have $|\lambda| < k$.

Since $L(G)$ is $(2k-2)$-regular, therefore $(2k-2)$ is the maximum eigenvalue of $L(G)$ with multiplicity one. This is a linkage between the maximum eigenvalues of G and $L(G)$. We can find a relationship between the spectra of G and $L(G)$ by means of the following theorem.

Theorem 2 *If G is a k-regular graph with n vertices and $m = \frac{1}{2}nk$ edges, then [7]*

$$\chi(L(G); \lambda) = (\lambda + 2)^{m-n}\chi(G; \lambda + 2 - k).$$

In other words, if the spectrum of G is

$$\mathrm{Spec}(G) = \begin{pmatrix} k & \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ 1 & o_1 & o_2 & \cdots & o_s \end{pmatrix},$$

then the spectrum of $L(G)$ is

$$\mathrm{Spec}(L(G)) =$$

$$\begin{pmatrix} k-2 & \overbrace{k-2+\lambda_1}^{\lambda_1^{(1)}} & \overbrace{k-2+\lambda_2}^{\lambda_2^{(1)}} & \cdots & \overbrace{k-2+\lambda_s}^{\lambda_s^{(1)}} & \overbrace{-2}^{\mu_1^1} \\ 1 & o_1 & o_2 & \cdots & o_s & m-n \end{pmatrix}.$$

Theorem 3 *If λ is an eigenvalue of a line graph $L(G)$, then $\lambda \geq -2$ [1].*

The condition that all eigenvalues of a graph be not less than -2 is a restrictive one, but it is not sufficient to characterize line graphs [8].

Now, we can consider $L(G)$ as main graph and obtain the spectrum of $L^2(G) = L(L(G))$ by means of the Theorem 2. First, we compute the number of vertices, edges and regularity of $L^2(G)$, namely $n^{(2)}, m^{(2)}$ and $k^{(2)}$. Similarly to the previous, a given edge e in $L(G)$ is adjacent with $(2k-3)$ edges in each of its endpoint, so $L^2(G)$ is $2(2k-3)$-regular and also

$$\begin{aligned} L^2(G) : n^{(2)} &= m^{(1)} = \frac{1}{2}nk(k-1) \\ m^{(2)} &= \frac{1}{2}n^{(2)}k^{(2)} = \frac{1}{2}nk(k-1)(2k-3) \\ k^{(2)} &= 2(2k-3) \end{aligned}$$

With respect to Theorem 2, the spectrum of $L^2(G)$ is

$$\mathrm{Spec}(L^2(G)) = \begin{pmatrix} 2(2k-3) & \overbrace{k^{(1)}-2+\lambda_1^{(1)}}^{\lambda_1^{(2)}} & \cdots \\ 1 & o_1 & \cdots \end{pmatrix}$$

$$\begin{pmatrix} \cdots & \overbrace{k^{(1)}-2+\lambda_s^{(1)}}^{\lambda_s^{(2)}} & \overbrace{k^{(1)}-2+\mu_1^{(1)}}^{\mu_1^{(2)}} & \overbrace{-2}^{\mu_2^2} \\ \cdots & o_s & m-n & m^{(1)}-n^{(1)} \end{pmatrix}$$

$$= \begin{pmatrix} 2(2k-3) & \overbrace{3(k-2)+\lambda_1}^{\lambda_1^{(2)}} & \overbrace{3(k-2)+\lambda_2}^{\lambda_2^{(2)}} & \cdots \\ 1 & o_1 & o_2 & \cdots \end{pmatrix}$$

$$\begin{pmatrix} \cdots & \overbrace{3(k-2)+\lambda_s}^{\lambda_s^{(2)}} & \overbrace{2(k-3)}^{\mu_1^{(2)}} & \overbrace{-2}^{\mu_2^2} \\ \cdots & o_s & m-n & m^{(1)}-n^{(1)} \end{pmatrix}.$$

Note that all eigenvalues of $L^2(G)$ satisfy the condition of the Theorem 3. Because, otherwise, there is an index (as l) such that $k^{(1)} + \lambda_l^{(1)} < 0$ or $k^{(1)} < 2$ that are contradiction (Without loss of generality, we can suppose $k > 2$, because a cycle is isomorphic to its line graph [9]). We can use the same argument iteratively and obtain a recurrence relation to find the number of vertices, edges and regularity of $L^t(G)$ for each positive integer t and then compute the spectrum of it.

Theorem 4 *If G is a k-regular graph with n vertices and $m = \frac{1}{2}nk$ edges and its spectrum is*

$$\mathrm{Spec}(G) = \begin{pmatrix} k & \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ 1 & o_1 & o_2 & \cdots & o_s \end{pmatrix},$$

then the spectrum of $L^t(G)$, for each positive integer t, is

$$\mathrm{Spec}\left(L^t(G)\right) = \begin{pmatrix} k^{(t)} & \lambda_1^{(t)} & \lambda_2^{(t)} & \cdots & \lambda_s^{(t)} & \mu_1^{(t)} & \cdots \\ 1 & o_1 & o_2 & \cdots & o_s & m-n \end{pmatrix}$$

$$\begin{pmatrix} \mu_2^{(t)} & \cdots & \mu_{t-1}^{(t)} & \mu_t^{(t)} \\ m^{(1)}-n^{(1)} & \cdots & m^{(t-1)}-n^{(t-1)} & -2 \end{pmatrix},$$

where for each $r = 1, 2, \cdots, t$

$$n^{(r)} = \frac{1}{2}nk \times \prod_{i=0}^{r-2} \left(2^i(k-2)+1\right)$$

$$m^{(r)} = \frac{1}{2}nk \times \prod_{i=0}^{r-1} \left(2^i(k-2)+1\right)$$

$$k^{(r)} = 2^r(k-2)+2$$

and

$$\lambda_l^{(t)} = \left(2^t - 1\right)(k-2) + \lambda_l; l = 1, \cdots s$$
$$\mu_l^{(t)} = 2^l \left(2^{t-1} - 1\right)(k-2) - 2; l = 1, \cdots t-1.$$

Proof We have used induction on r to prove formulas specifying the number of vertices and regularity of $L^r(G); r = 1, \cdots, t$. Regularity: Basis step: for $r = 1, 2$, the formula holds. Because: $k^{(1)} = 2(k-1) = 2^1(k-2) + 2, k^{(2)} = 2\left(k^{(1)} - 1\right) = 2(2k-3) = 2^2(k-2) + 2$. Induction step: We suppose that the formula holds for $k^{(r-1)}$, that is $k^{(r-1)} = 2^{r-1}(k-2) + 2$. Therefore

$$k^{(r)} = 2\left(k^{r-1} - 1\right) = 2\left[2^{r-1}(k-2) + 2 - 1\right] = 2^r(k-2) + 2.$$

The number of vertices: Basis step: for $r = 2$, the formula holds. Because

$$n^{(2)} = \frac{1}{2}nk(k-1) = \frac{1}{2}nk \times \Pi_{i=0}^{0}\left(2^i(k-2) + 1\right).$$

Induction step: We suppose that the formula holds for $n^{(r-1)}$, that is

$$n^{(r-1)} = \frac{1}{2}nk \times \Pi_{i=0}^{r-3}\left(2^i(k-2) + 1\right).$$

Therefore

$$\begin{aligned} n^{(r)} &= m^{(r-1)} = \frac{1}{2}n^{(r-1)}k^{(r-1)} \\ &= \frac{1}{2}\left[\frac{1}{2}nk \times \Pi_{i=0}^{r-3}\left(2^i(k-2)+1\right)\right]\left[2^{r-1}(k-2)+2\right] \\ &= \frac{1}{2}nk \times \Pi_{i=0}^{r-2}\left(2^i(k-2)+1\right). \end{aligned}$$

The number of edges

$$\begin{aligned} m^{(r)} &= \frac{1}{2}n^{(r)}k^{(r)} = \frac{1}{2}\left[\frac{1}{2}nk \times \Pi_{i=0}^{r-2}\left(2^i(k-2)+1\right)\right]\left[2^r(k-2)+2\right] \\ &= \frac{1}{2}nk \times \Pi_{i=0}^{r-1}\left(2^i(k-2)+1\right). \end{aligned}$$

Now, we compute the eigenvalues of $L^t(G)$. Using Theorem 2, for each $l = 1, \cdots s$, we have

$$\begin{aligned} \lambda_l^{(t)} &= k^{(t-1)} - 2 + \lambda_l^{(t-1)} = k^{(t-1)} - 2 + \left(k^{(t-2)} - 2 + \lambda_l^{(t-2)}\right) \\ &= k^{(t-1)} - 2 + \left(k^{(t-2)} - 2 + \left(k^{(t-3)} - 2 + \lambda_l^{(t-3)}\right)\right) \\ &= k^{(t-1)} - 2 + \left(k^{(t-2)} - 2 + \left(k^{(t-3)} - 2 + \left(\cdots + \left(k^{(1)}\right.\right.\right.\right. \\ &\quad -2 + (k - 2 + \lambda_l)))) \\ &= k^{(t-1)} + k^{(t-2)} + k^{(t-3)} + \cdots + k^{(1)} + k - 2t + \lambda_l. \end{aligned}$$

$k^{(r)}; r = 1, \cdots, t-1$ can be counted using the regularity formula, so

$$\begin{aligned}\lambda_l^{(t)} &= \left[2^{t-1}(k-2)+2\right] + \left[2^{t-2}(k-2)+2\right] \\ &+ \cdots + \left[2^1(k-2)+2\right] + k - 2t + \lambda_l \\ &= \left(2^t - 1\right)(k-2) + \lambda_l.\end{aligned}$$

Finally, we prove the formula specifying $\mu_l^{(t)};\ l = 1, \cdots, t-1$ using induction on t. Basis step: for $t = 2$ (so $l = 1$), the formula holds. Because

$$\mu_1^{(2)} = k^{(1)} - 2 - 2 = 2k - 6 = 2^1\left(2^{2-1} - 1\right)(k-2) - 2.$$

Induction step: We suppose that the formula holds for $\mu_l^{(t-1)}; l = 1, \cdots, t-2$, that is

$$\mu_l^{(t-1)} = 2^l\left(2^{t-2} - 1\right)(k-2) - 2; l = 1, \cdots, t-2.$$

Therefore

$$\begin{aligned}\mu_l^{(t)} &= k^{(t-1)} - 2 + \mu_l^{(t-1)} \\ &= 2^{t-1}(k-2) + 2 - 2 + 2^l\left(2^{t-l-1} - 1\right)(k-2) - 2 \\ &= 2^l\left(2^{t-l-1} + 2^{t-l-1} - 1\right)(k-2) - 2 \\ &= 2^l\left(2^{t-1} - 1\right)(k-2) - 2; l = 1, \cdots, t-1.\end{aligned}$$

Corollary 1 *If G is a k-regular graph with n vertices and $m = \frac{1}{2}nk$ edges and its spectrum is*

$$\mathrm{Spec}(G) = \begin{pmatrix} k & \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ 1 & o_1 & o_2 & \cdots & o_s \end{pmatrix}$$

and the spectrum of $L^t(G)$, for each positive integer t, is

$$\mathrm{Spec}\left(L^t(G)\right) = \begin{pmatrix} k^{(t)} & \lambda_1^{(t)} & \lambda_2^{(t)} & \cdots & \lambda_s^{(t)} & \mu_1^{(t)} & \cdots \\ 1 & o_1 & o_2 & \cdots & o_s & m-n & \\ & \mu_2^{(t)} & \cdots & \mu_{t-1}^{(t)} & \mu_t^{(t)} \\ & m^{(1)} - n^{(1)} & \cdots & m^{(t-1)} - n^{(t-1)} & -2 \end{pmatrix},$$

then

$$\begin{aligned}k^{(t)} - \lambda_1^{(t)} &= k - \lambda_1 \\ \lambda_l^{(t)} - \lambda_{l+1}^{(t)} &= \lambda_l - \lambda_{l+1}; l = 1, \cdots, s-1, \\ \lambda_s^{(t)} - \mu_1^{(t)} &= k + \lambda_s \\ \mu_l^{(t)} - \mu_{l+1}^{(t)} &= 2^l(k-2); l = 1, \cdots, t-1\end{aligned}$$

Finally, suppose G be the complete graph with 4 vertices. However, this graph is the smallest regular graph ($k > 2$), it is difficult to draw $L^t(G); t \geq 3$, for this graph and to get its spectrum. Using the Theorem 4, we have

$$\begin{aligned}L^{10}(K_4) : n^{(10)} &= 1,958,457,114,900 \\ m^{(10)} &= 1.00468850E + 15 \\ k^{(10)} &= 1026\end{aligned}$$

Also, we compute the spectrum of $L^t(K_4)\,; t = 1, \cdots, 10$. The results are shown in Table 1.

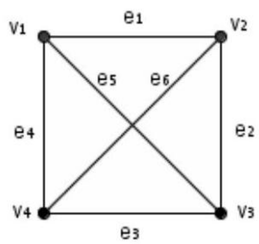

Fig. 1 G $(n = 4, m = 6, k = 3)$

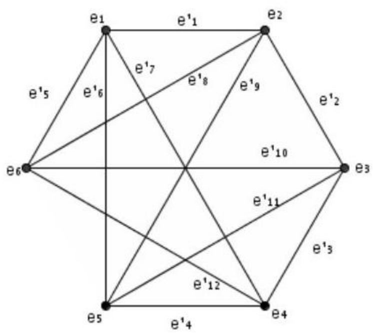

Fig. 2 $L(G)$ $(n' = 6, m' = 12, k' = 4)$

3 Conclusion and future work

Most of the results obtained in this work are based on the study of simple, undirected, and finite graphs. Further studies are to be done and appropriate proof techniques are to be employed to generalize and direct graphs. However, the results can be obtained for every graph.

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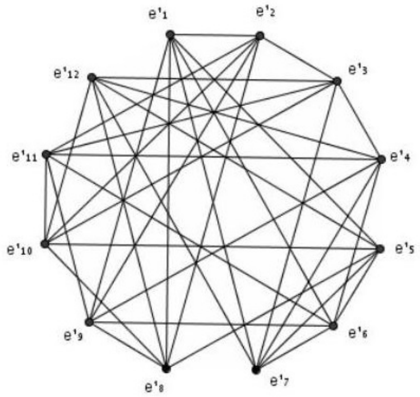

Fig. 3 $L^2(G)$ $(n^{(2)} = 12, m^{(2)} = 36, k^{(2)} = 6)$

Table 1: the eigenvalues of $L^t(K_4); t = 1, 2, \ldots, 10$ and their multiplicities

	$k^{(0)}$	$\lambda_1^{(0)}$	$\mu_1^{(0)}$	$\mu_2^{(0)}$	$\mu_3^{(0)}$	$\mu_4^{(0)}$	$\mu_5^{(0)}$	$\mu_6^{(0)}$	$\mu_7^{(0)}$	$\mu_8^{(0)}$	$\mu_9^{(0)}$	$\mu_{10}^{(0)}$
K_4	3	-1										
$L(K_4)$	4	0	-2									
$L^2(K_4)$	6	2	0	-2								
$L^3(K_4)$	10	6	4	2	-2							
$L^4(K_4)$	18	14	12	10	6	-2						
$L^5(K_4)$	34	30	28	26	22	14	-2					
$L^6(K_4)$	66	62	60	58	54	46	30	-2				
$L^7(K_4)$	130	126	124	122	118	110	94	62	-2			
$L^8(K_4)$	258	254	252	250	246	238	222	190	126	-2		
$L^9(K_4)$	514	510	508	506	502	494	478	446	382	254	-2	
$L^{10}(K_4)$	1026	1022	1020	1018	1014	1006	990	958	894	766	510	-2
multiplcity	1	3	2	6	24 144	1440	25920	881280	58164480	7561382400	1950836659200	

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