Global Analysis and Discrete Mathematics
Volume x, Issue x, pp. xx–xx
ISSN: 2476-5341

A Survey on Existence of a Solution to Singular Fractional Difference Equation

Mohsen Khaleghi Moghadam

Received: 20 December 2023/ Accepted: 5 February 2024

Abstract In this paper, we deal with the existence of a positive solution for the following fractional discrete boundary-value problem

$$\begin{cases} {}_{T+1}\nabla_k^{\alpha}\left({}_k\nabla_0^{\alpha}(u(k))\right) = \lambda f(k, u(k)), & k \in [1, T]_{\mathbb{N}_0}, \\ u(0) = u(T+1) = 0, \end{cases}$$

where $0 < \alpha < 1$ and ${}_k\nabla_0^{\alpha}$ is the left nabla discrete fractional difference and ${}_{T+1}\nabla_k^{\alpha}$ is the right nabla discrete fractional difference $f : [1, T]_{\mathbb{N}_0} \times (0, +\infty) \to \mathbb{R}$ may be singular at $t = 0$ and may change sign and $\lambda > 0$ is a parameter. The technical method is variational approach for differentiable functionals. An example is included to illustrate the main results.

Keywords Discrete fractional calculus · Discrete nonlinear boundary value problem · Non trivial solution · Variational methods · Critical point theory

Mathematics Subject Classification (2010) 26A33 · 39A10 · 39A27

ARTICLE IN PRESS

1 Introduction

Initial and boundary value problems in discrete fractional calculus play a fundamental role in different fields of research, for example in Biology, Atici and Şengül introduced and solved Gompertz fractional difference equation for tumor growth models [5]. In [6] authors studied multiple positive solutions of singular discrete boundary value problems using variational methods. It is well known that variational methods is an important tool to deal with the problems for differential and difference equations. Variational methods for dealing

M. Khaleghi Moghadam
Department of Basic Sciences, Sari Agricultural Sciences and Natural Resources University, Sari, Iran.
E-mail: mohsen.khaleghi@rocketmail.com, m.khaleghi@sanru.ac.ir

with fractional difference equations with boundary value conditions have appeared in [10]. More, recently, in [9,13–16], the existence and multiplicity of solutions for nonlinear discrete boundary value problems have been investigated by adopting variational methods. In [7] authors employed the critical point theory to establish the existence of multiple solutions of some regular as well as singular discrete boundary value problems. Other results on discrete boundary value problems can be found in [4,17] in the nonsingular case and in [18–20,11] in the singular case. The first concepts of fractional nabla differences traces back to the works of Gray and Zhang [23].

We refer the reader to the new monograph [26] that works for differential and integral equations and systems and for many theoretical and applied problems in mathematics, mathematical physics, probability and statistics, applied computer science and numerical methods. Also we refer the reader to the recent monograph on the introduction to fractional nabla calculus [12]. Another well-known monograph is [24] that is devoted to the systematic and comprehensive exposition of classical and modern results in the theory of fractional integrals and derivatives and their applications.

There seems to be increasing interest in the existence of solutions to boundary value problems for finite difference equations with fractional difference operator during the last two decades. In last decades, some researchers investigated q-fractional difference equations. Later, q-fractional boundary value problems considered by many researchers; see for instance, [25] and references therein. The other important tool in the study of nonlinear difference equations is upper and lower solution method; see, for instance, [18,11] and references therein. Morse theory is also other tool in the study of nonlinear fractional differential equations [21].

The aim of this paper is to establish the existence of non-trivial solution for the following discrete boundary-value problem

$$\begin{cases} {}_{T+1}\nabla_k^\alpha \left({}_k\nabla_0^\alpha(u(k))\right) = \lambda f(k,u(k)), \quad k \in [1,T]_{\mathbb{N}_0}, \\ u(0) = u(T+1) = 0, \end{cases} \tag{1}$$

where $0 < \alpha < 1$ and ${}_k\nabla_0^\alpha$ is the left nabla discrete fractional difference and ${}_{T+1}\nabla_k^\alpha$ is the right nabla discrete fractional difference and

$$\nabla u(k) = u(k) - u(k-1),$$

is the backward difference operator and $\lambda > 0$ is a parameter and $T \geq 2$ is fixed positive integer and $\mathbb{N}_1 = \{1,2,3,\dots\}$ and ${}_T\mathbb{N} = \{\dots,T-2,T-1,T\}$ and $[1,T]_{\mathbb{N}_0}$ is the discrete set $\{1,2,\dots,T-1,T\} = \mathbb{N}_1 \bigcap {}_T\mathbb{N}$ and we only assume that $f : [1,T]_{\mathbb{N}_0} \times (0,\infty) \to \mathbb{R}$ satisfies

$$a_0(k) \leq f(k,t) \leq a_1(k)t^{-\gamma}, \qquad (k,t) \in [1,T]_{\mathbb{N}_0} \times (0,t_0) \tag{2}$$

for some nontrivial functions $a_0, a_1 > 0$ and $\gamma, t_0 > 0$, so that it may be singular at $t = 0$ and may change sign.

In this paper we ensure an exact interval of the parameter λ, in which the problem (1) admits at least a positive solution. Here, we point out the following our main result.

ARTICLE IN PRESS

Theorem 1 *If* (2) *holds and*

$$\limsup_{t\to\infty} \frac{f(k,t)}{t} < \lambda_1, \quad k \in [1,T]_{\mathbb{N}_0}, \tag{3}$$

then (1) *has a positive solution for any* $\lambda \in \left[\limsup_{t\to\infty} \frac{f(k,t)}{t}, \lambda_1\right)$.

The rest of this paper is arranged as follows. In Section 2, we provide some basic definitions and preliminary results and fundamental functionals and lemmas. In Section 3, we provide our main results and finally, we illustrate the result by giving example.

2 Preliminaries

The following definitions will be helpful for our discussion.

Definition 1 [3] **(i)** Let m be a natural number, then the m rising factorial of t is written as

$$t^{\overline{m}} = \prod_{k=0}^{m-1} (t+k), \qquad t^{\overline{0}} = 1. \tag{4}$$

(ii) For any real number, the α rising function is increasing on $\mathbb{N}_0$ and

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \qquad \text{such that} \quad t \in \mathbb{R}\backslash\{\ldots,-2,-1,0\}, \quad 0^{\overline{\alpha}} = 0. \tag{5}$$

Definition 2 Let f be defined on $\mathbb{N}_{a-1} \bigcap {}_{b+1}\mathbb{N}$, $a < b$, $\alpha \in (0,1)$, then the nabla discrete new (left Gerasimov-Caputo) fractional difference is defined by

$$\left({}_{k}^{C}\nabla_{a-1}^{\alpha} f\right)(k) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=a}^{k} \nabla_s f(s)(k-\rho(s))^{\overline{-\alpha}}, \quad k \in \mathbb{N}_a, \tag{6}$$

and the right Gerasimov-Caputo one by

$$\left({}_{b+1}^{C}\nabla_{k}^{\alpha} f\right)(k) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=k}^{b} (-\Delta_s f)(s)(s-\rho(k))^{\overline{-\alpha}}, \quad k \in {}_{b}\mathbb{N}, \tag{7}$$

and in the left Riemann-Liouville sense by

$$\begin{aligned} \left({}_{k}^{R}\nabla_{a-1}^{\alpha} f\right)(k) &= \frac{1}{\Gamma(1-\alpha)} \nabla_k \sum_{s=a}^{k} f(s)(k-\rho(s))^{\overline{-\alpha}}, \quad k \in \mathbb{N}_a, \\ &= \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{k} f(s)(k-\rho(s))^{\overline{-\alpha-1}}, \quad k \in \mathbb{N}_a, \end{aligned} \tag{8}$$

ARTICLE IN PRESS

and the right Riemann-Liouville one by

$$
\begin{aligned}
\left({}_{b+1}^{\;\;R}\nabla_k^\alpha f\right)(k) &= \frac{1}{\Gamma(1-\alpha)}(-\Delta_k)\sum_{s=k}^{b} f(s)(s-\rho(k))^{\overline{-\alpha}}, \quad k \in {}_b\mathbb{N}, \\
&= \frac{1}{\Gamma(-\alpha)}\sum_{s=k}^{b} f(s)(s-\rho(k))^{\overline{-\alpha-1}}, \quad k \in {}_b\mathbb{N},
\end{aligned} \tag{9}
$$

where $\rho(k) = k-1$ be the backward jump operator.

For example, Let $f(k) = 1$ be defined on $\mathbb{N}_{a-1}\bigcap {}_{b+1}\mathbb{N}$, therefore from (6) and (7), we have

$$
{}_{b+1}^{\;\;C}\nabla_k^\alpha 1 = {}_k^C\nabla_{a-1}^\alpha 1 = 0, \quad k \in \mathbb{N}_a \bigcap {}_b\mathbb{N}. \tag{10}
$$

The relation between the nabla left and right Gerasimov-Caputo and Riemann-Liouville fractional differences are as follow:

$$
\left({}_k^C\nabla_{a-1}^\alpha f\right)(k) = \left({}_k^R\nabla_{a-1}^\alpha f\right)(k) - \frac{(k-a+1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} f(a-1), \tag{11}
$$

$$
\left({}_{b+1}^{\;\;C}\nabla_k^\alpha f\right)(k) = \left({}_{b+1}^{\;\;R}\nabla_k^\alpha f\right)(k) - \frac{(b+1-k)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} f(b+1). \tag{12}
$$

Thus by (10), (11), and (12), we have for any $k \in \mathbb{N}_a \bigcap {}_b\mathbb{N}$,

$$
{}_{b+1}^{\;\;R}\nabla_k^\alpha 1 = \frac{(b+1-k)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}, \quad {}_k^R\nabla_{a-1}^\alpha 1 = \frac{(k-a+1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}. \tag{13}
$$

Regarding the domains of the fractional type differences we observe:

(i) The nabla left fractional difference ${}_k\nabla_{a-1}^\alpha$ maps functions defined on ${}_{a-1}\mathbb{N}$ to functions defined on ${}_a\mathbb{N}$.
(ii) (The nabla right fractional difference ${}_{b+1}\nabla_k^\alpha$ maps functions defined on ${}_{b+1}\mathbb{N}$ to functions defined on ${}_b\mathbb{N}$.

As in [8] one can show that, for $\alpha \to 0$, one has ${}_k\nabla_a^\alpha(f(k)) \to f(t)$ and for $\alpha \to 1$, one has ${}_k\nabla_a^\alpha(f(k)) \to \nabla f(t)$. We note that the nabla Riemann-Liouville and Gerasimov-Caputo fractional differences, for $0 < \alpha < 1$, coincide when f vanishes at the end points, that is, $f(a-1) = 0 = f(b+1)$ (see [1]). Indeed, when $0 < \alpha < 1$, those conclude from (11) and (12). So, for convenience, from now on we will use the symbol ∇^α instead of ${}^R\nabla^\alpha$ or ${}^C\nabla^\alpha$. Let $\lambda_1 > 0$ be the first and smallest eigenvalue and $\lambda_2 > 0$ be the last and biggest eigenvalue of

$$
\begin{cases}
{}_{T+1}\nabla_k^\alpha\left({}_k\nabla_0^\alpha(u(k))\right) = \lambda u(k), \quad k \in [1, T]_{\mathbb{N}_0}, \\
u(0) = u(T+1) = 0,
\end{cases} \tag{14}
$$

ARTICLE IN PRESS

where

$$\lambda_1 = \min_{u \in W \setminus \{0\}} \frac{u^\dagger \mathbb{A} u}{u^\dagger u} = \min_{u \in W \setminus \{0\}} \frac{\sum_{k=1}^{T} |\left({}_k\nabla_0^\alpha u\right)(k)|^2}{\sum_{k=1}^{T} |u(k)|^2},$$

$$\lambda_2 = \max_{u \in W \setminus \{0\}} \frac{u^\dagger \mathbb{A} u}{u^\dagger u} = \max_{u \in W \setminus \{0\}} \frac{\sum_{k=1}^{T} |\left({}_k\nabla_0^\alpha u\right)(k)|^2}{\sum_{k=1}^{T} |u(k)|^2},$$

which $u^\dagger$ denotes the transpose of u and $\frac{u^\dagger \mathbb{A} u}{u^\dagger u}$ is called a Rayleigh quotient and $\mathbb{A}$ is called a matrix structure form of operator ${}_{T+1}\nabla_k^\alpha \left({}_k\nabla_0^\alpha (u(k))\right)$. Taking the definition $\mathbb{A}$ into account, (14) converts to

$$\mathbb{A} \begin{bmatrix} u(1) \\ u(2) \\ u(3) \\ \vdots \\ u(T) \end{bmatrix} = \lambda \begin{bmatrix} u(1) \\ u(2) \\ u(3) \\ \vdots \\ u(T) \end{bmatrix}.$$

where $\mathbb{A} := C^\dagger C$ and

$$C = \begin{bmatrix} \gamma_1 & 0 & 0 & \cdots & 0 \\ \gamma_2 & \gamma_1 & 0 & \cdots & 0 \\ \gamma_3 & \gamma_2 & \gamma_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_T & \gamma_{T-1} & \gamma_{T-2} & \cdots & \gamma_1 \end{bmatrix}$$

By (4), $\gamma_i := \frac{-\alpha \Gamma(i-1-\alpha)}{\Gamma(i).\Gamma(1-\alpha)}$ for any $i = 1, 2, 3, 4, \ldots, T$. It is clear that $\gamma_1 = 1$ and $\gamma_2 = -\alpha$ and $\gamma_{i+1} = \frac{i-1-\alpha}{i}\gamma_i$ and $-\alpha < \gamma_{i-1} < \gamma_i < 0$ for any $i = 3, 4, \ldots, T$.

Remark 1 The matrix $C = [c_{ij}]$ has all the elements below the main diagonal as negative. The diagonal elements of the matrix C are 1, that is, $c_{ii} = 1$, $i = 1, 2, \ldots, T$ and the determinant of the matrix $C = [c_{ij}]$ where denoted by the symbol $|C|$ is 1 and then that is non-singular or invertible.

Remark 2 ([22]) The matrix C^{-1} is lower triangular and it is obtained from the identity matrix I_n by performing the same sequence of elementary row operations as were used to convert C^{-1} to I_n. By elementary row operations method, the diagonal elements of the matrix $C^{-1} = [c_{ij}]$ are 1 and it has all the elements below the main diagonal as positive.

So, the matrix $\mathbb{A} = C^\dagger C$ is real symmetric matrixes and by Remark 1, the determinant of the matrix $\mathbb{A}$ is 1 and $\mathbb{A}$ is invertible and the determinant of the matrix $\mathbb{A}^{-1}$ is 1.

Taking the definition $\mathbb{A}$ into account, (1) converts to

$$\mathbb{A} \begin{bmatrix} u(1) \\ u(2) \\ u(3) \\ \vdots \\ u(T) \end{bmatrix} = \lambda \begin{bmatrix} f(1, u(1)) \\ f(2, u(2)) \\ f(3, u(3)) \\ \vdots \\ f(T, u(T)) \end{bmatrix}.$$

ARTICLE IN PRESS

Now, we present summation by parts formula in new discrete fractional calculus.

Theorem 2 *([2, Theorem 4.4] Integration by parts for fractional difference) For functions f and g defined on $\mathbb{N}_a \bigcap {}_b\mathbb{N}$, $a \equiv b$ (mod 1), and $0 < \alpha < 1$, one has*

$$\sum_{k=a}^{b} f(k)\left({}_k\nabla_{a-1}^{\alpha} g\right)(k) = \sum_{k=a}^{b} g(k)\left({}_{b+1}\nabla_{k}^{\alpha} f\right)(k). \tag{15}$$

Similarly,

$$\sum_{k=a}^{b} f(k)\left({}_{b+1}\nabla_{k}^{\alpha} g\right)(k) = \sum_{k=a}^{b} g(k)\left({}_k\nabla_{a-1}^{\alpha} f\right)(k). \tag{16}$$

3 Main Results

In order to give the variational formulation of the problem (1), let us define the finite $T-$dimensional Hilbert space

$$W := \{u : [0, T+1]_{\mathbb{N}_0} \to \mathbb{R} : u(0) = u(T+1) = 0\},$$

which W is equipped with the usual inner product and the norm

$$\langle u, v\rangle = \sum_{k=1}^{T} u(k)v(k), \qquad \|u\|_2 := \left(\sum_{k=1}^{T} |u(k)|^2\right)^{\frac{1}{2}}.$$

It is known that the following norm

$$\|u\| = \left\{\sum_{k=1}^{T} \left|\left({}_k\nabla_0^{\alpha} u\right)(k)\right|^2\right\}^{\frac{1}{2}},$$

is an equivalent norm in W. It is clear that λ_1 and λ_2 are respectively the first and the last eigenvalues of $\mathbb{A}$ and

$$\lambda_1 = \min_{u\in W\setminus\{0\}} \frac{\|u\|^2}{\|u\|_2^2}, \qquad \lambda_2 = \max_{u\in W\setminus\{0\}} \frac{\|u\|^2}{\|u\|_2^2}.$$

So for any $u \in W$, we have,

$$\sqrt{\lambda_1}\|u\|_2 < \|u\| < \sqrt{\lambda_2}\|u\|_2. \tag{17}$$

Therefor from (17), $\|u\| \to +\infty$ if and only if $\|u\|_2 \to +\infty$.
Let $\Phi : W \to \mathbb{R}$ be the functional

$$\Phi(u) := \frac{1}{2}\sum_{k=1}^{T} \left|\left({}_k\nabla_0^{\alpha} u\right)(k)\right|^2. \tag{18}$$

ARTICLE IN PRESS

An easy computation ensures that Φ turns out to be of class C^1 on W and Gateaux differentiable with

$$\Phi'(u)(v) = \sum_{k=1}^{T} \left({}_k\nabla_0^\alpha(u(k))\right)\left({}_k\nabla_0^\alpha v(k)\right), \qquad \text{for all } u, v \in W.$$

To study the problem (1), for every $\lambda > 0$, we consider the functional $I_{\lambda,g} : W \to \mathbb{R}$ defined by

$$I_{\lambda,g}(u) := \Phi(u) - \lambda\Psi(u), \quad \Psi(u) := \sum_{k=1}^{T} G(k, u(k)), \tag{19}$$

where $G(k,u) = \int_0^u g(k,t)dt$ where $g \in C([1,T]_{\mathbb{N}_0} \times \mathbb{R})$.

Lemma 1 *The function u be a critical point of $I_{\lambda,g}$ in W, iff u be a solution of the problem* (1).

For the approach of Lemma 1 see [13, Lemma 3.1].

Lemma 2 *If*

$$\limsup_{t\to\infty} \frac{g(k,t)}{t} < \lambda_1, \quad k \in [1,T]_{\mathbb{N}_0}, \tag{20}$$

holds, then $I_{\lambda,g}$ has a global minimizer for any $\lambda \in [\limsup_{t\to\infty} \frac{g(k,t)}{t}, \lambda_1)$.

Proof By (20), for any $\lambda \in [\limsup_{t\to\infty} \frac{g(k,t)}{t}, \lambda_1)$, there exists $1 < r_2$ such that

$$G(k,t) \le \frac{\lambda}{2}|t|^2, \quad \text{for any } (k,t) \in [1,T] \times \mathbb{R} \setminus ([-r_2, r_2]).$$

Since G is continuous, then $G(k,t)$ is bounded for any $(k,t) \in [1,T] \times [-r_2, r_2]$, so we can choose $C > 0$ such that

$$G(k,t) \le \frac{\lambda}{2}|t|^2 + C, \quad \text{for all } (k,t) \in [1,T] \times \mathbb{R}, \tag{21}$$

ARTICLE IN PRESS

by (17) and (21), we have

$$\begin{aligned}
I_{\lambda,g}(u) &= \Phi(u) - \lambda\Psi(u)\\
&= \frac{1}{2}\sum_{k=1}^{T} |\left({}_k\nabla_0^\alpha u\right)(k)|^2 - \lambda\sum_{k=1}^{T} G(k,u(k))\\
&= \frac{1}{2}\|u\|^2 - \lambda\sum_{k=1}^{T} G(k,u(k))\\
&\geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\sum_{k=1}^{T} |u(k)|^2 - TC\\
&= \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u\|_2^2 - TC\\
&\geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{2\lambda_1}\|u\|^2 - TC\\
&= \frac{1}{2}(1-\frac{\lambda}{\lambda_1})\|u\|^2 - TC,
\end{aligned}$$

therefore as $\|u\| \to +\infty$, $I_{\lambda,g}(u) \to +\infty$. Hence $I_{\lambda,g}$ is coercive and bounded from below for any $\lambda \in [\limsup_{t\to\infty} \frac{g(k,t)}{t}, \lambda_1)$. So $I_{\lambda,g}$ has a global minimizer for any $\lambda \in [\limsup_{t\to\infty} \frac{g(k,t)}{t}, \lambda_1)$ (see [27]).

We obtain the next result which guarantees the same conclusion of the Strong maximum principle.

Lemma 3 *If $\lambda > 0$ and $u \in W$ be a non-trivial solution to problem*

$$\begin{cases} {}_{T+1}\nabla_k^\alpha\left({}_k\nabla_0^\alpha(u(k))\right) = \lambda a_0(k), & k \in [1,T]_{\mathbb{N}_0},\\ u(0) = u(T+1) = 0, \end{cases} \tag{22}$$

then $u > 0$.

Proof By taking account Remark 1, one can conclude that $|\mathbb{A}| = |C||C^\dagger| = 1$, hence the matrix $\mathbb{A}$ is non-singular or invertible and the equation (22) becomes

$$\begin{bmatrix} u(1)\\ u(2)\\ u(3)\\ \vdots\\ u(T)\end{bmatrix} = \lambda\mathbb{A}^{-1}\begin{bmatrix} a_0(1)\\ a_0(2)\\ a_0(3)\\ \vdots\\ a_0(T)\end{bmatrix},$$

where $\mathbb{A}^{-1} = [a_{ij}]$, $i,j \in [1,T]_{\mathbb{N}_0}$ is the inverse of $\mathbb{A}$ and $\mathbb{A}^{-1} = C^{-1}\{C^{-1}\}^\dagger$. Taking account Remark 2, C^{-1} has all the elements below the main diagonal as positive and $\{C^{-1}\}^\dagger$ has all the elements above the main diagonal as positive. So $\mathbb{A}^{-1}$ has all the elements as positive. Since $\lambda > 0$ and $a_0(k) > 0$ for every $k \in [1,T]_{\mathbb{N}_0}$ and $a_{ij} > 0$, $i,j \in [1,T]_{\mathbb{N}_0}$, one can conclude that

$$u(k) = \lambda(a_{k1} + a_{k2} + \cdots + a_{kT})a_0(k) > 0, \quad \text{for every } k \in [1,T]_{\mathbb{N}_0}.$$

ARTICLE IN PRESS

Here, we can provide the proof of main result of this paper (Theorem 1) which that was stated before in Introduction section.

Proof Since $a_0 \in C([1,T]_{\mathbb{N}_0} \times \mathbb{R})$ and

$$\limsup_{t\to\infty} \frac{a_0(k)}{t} = 0 < \lambda_1, \quad k \in [1,T]_{\mathbb{N}_0},$$

by Lemma 2 the functional I_{λ,a_0} has a global minimizer for any $\lambda \in [0,\lambda_1)$, so by Lemma 1, the problem (22) has a solution $u_0 \in W$ for any $\lambda \in [0,\lambda_1)$ also since $a_0(k) > 0$ for any $k \in [1,T]_{\mathbb{N}_0}$ by Lemma 3 this solution is positive. Fix $\varepsilon \in (0,1]$ small enough such that $\underline{u} := \varepsilon u_0 < t_0$. Then by (2)

$$_{T+1}\nabla_k^{\alpha}\left(_k\nabla_0^{\alpha}(\underline{u}(k))\right) = \lambda\varepsilon a_0(k) \le \lambda\varepsilon f(k,\underline{u}(k)) \le \lambda f(k,\underline{u}(k)), \tag{23}$$

so $\underline{u}$ is a sub-solution of (1). Let

$$f_{\underline{u}}(k,t) = \begin{cases} f(k,t), & t \ge \underline{u}(k), \\ f(k,\underline{u}(k)), & t < \underline{u}(k). \end{cases} \tag{24}$$

so $f_{\underline{u}}(k,t) \in C([1,T]_{\mathbb{N}_0} \times \mathbb{R})$. By similar argument in [6], due to (3), there are $\lambda \in (0,\lambda_1)$ and $T > t_0$ such that $f(k,t) \le \lambda t$ for any $(k,t) \in [1,T]_{\mathbb{N}_0} \times (T,\infty)$. Then taking account to (2), one can conclude that

$$f_{\underline{u}}(k,t) \le a_1(k)\underline{u}^{-\gamma} + \max f([1,T]_{\mathbb{N}_0} \times [t_0,T]) + \lambda t, \quad \forall t \ge 0, \tag{25}$$

and then $\limsup_{t\to\infty} \frac{f_{\underline{u}}(k,t)}{t} \le \lambda < \lambda_1$ and due to $f_{\underline{u}} \in C([1,T]_{\mathbb{N}_0} \times \mathbb{R})$, by Lemma 2, the modified problem

$$\begin{cases} _{T+1}\nabla_k^{\alpha}\left(_k\nabla_0^{\alpha}(u(k))\right) = \lambda f_{\underline{u}}(k,u(k)), & k \in [1,T]_{\mathbb{N}_0}, \\ u(0) = u(T+1) = 0, \end{cases} \tag{26}$$

has a solution u for any $\lambda \in [\limsup_{t\to\infty} \frac{f_{\underline{u}}(k,t)}{t}, \lambda_1) = [\limsup_{t\to\infty} \frac{f(k,t)}{t}, \lambda_1)$. If $u < \underline{u}$ by (26) and (24) and (23), one has

$$\begin{aligned} _{T+1}\nabla_k^{\alpha}\left(_k\nabla_0^{\alpha}(u(k))\right) &= \lambda f_{\underline{u}}(k,u(k)) = \lambda f(k,\underline{u}(k)) \\ &\ge {}_{T+1}\nabla_k^{\alpha}\left(_k\nabla_0^{\alpha}(\underline{u}(k))\right), \end{aligned}$$

so, $_{T+1}\nabla_k^{\alpha}\left(_k\nabla_0^{\alpha}(u - \underline{u}(k))\right) \ge 0$ for $k \in [1,T]_{\mathbb{N}_0}$ and then by Lemma 3, $u \ge \underline{u}$ and it is a contradiction. So $u \ge \underline{u}$ and then by (26), one can conclude that u is a solution of (1) for any $\lambda \in [\limsup_{t\to\infty} \frac{f(k,t)}{t}, \lambda_1)$ and due to $u \ge \underline{u} > 0$, that is positive.

We now present an example to illustrate the result of Theorem 1.

Example 1 the singular fractional problem

$$\begin{cases} _{T+1}\nabla_k^{\alpha}\left(_k\nabla_0^{\alpha}(u(k))\right) = \frac{\lambda}{u^2(k)}, & k \in [1,9], \\ u(0) = u(10) = 0, \end{cases}$$

has at least one positive solution u_0 for any $\lambda \in [0,\lambda_1)$, since the function $f(k,t) = \frac{1}{t^2}$ satisfies (2) and $\limsup\limits_{t\to+\infty} \frac{f(k,t)}{t} = 0$.

ARTICLE IN PRESS

4 Conclusion

Most phenomena in the nature could be modeled by different types of fractional difference and differential equations with initial or boundary conditions. Recently some scientists have been studying the role of fractional calculus in better describing of physical phenomena or biological phenomena. They have found that by using the fractional difference equations they can provide a better representation by some physical or biological concepts. Thus, we should investigate distinct fractional difference equations to increase our ability for exact modelings of more phenomena. In this work, by using variational method on differentiable functionals, we study a singular fractional problem with Dirichlet boundary value conditions. We provide an example to illustrate our main result.

Acknowledgements

The author is thankful to the reviewers and the editor for their comments which have improved the quality of the article.

Funding

This work was supported by Sari agricultural sciences and natural resources university [grant number 03-1403-01].

ARTICLE IN PRESS

References

1. T. Abdeljawad, On delta and nabla Caputo fractional differences and dual identities, Discrete Dynamics in Nature and Society, 2013, (2013).
2. T. Abdeljawad, F. Atici, On the Definitions of Nabla Fractional Operators, Abstract and applied Analysis, 2012, 13 pages (2012).
3. F. M. Atici, PW. Eloe, Discrete fractional calculus with the nabla operator, Electron. J. Qual. Theory Differ. Equ. Spec. Ed. I, 2009, 3 (2009).
4. R. Avery, J. Henderson, Existence of three positive pseudo-symmetric solutions for a one dimensional discrete p-Laplacian, J. Difference Equ. Appl., 10, 529–539 (2004).
5. F. M. Atici, S. Şengül, Modeling with fractional difference equations, J. Math. Anal. Appl., 369, 1–9 (2010).
6. R. P. Agarwal, K. Perera, D. O'Regan, Multiple positive solutions of singular discrete p-Laplacian problems via variational methods, Adv. Diff. Equ., 2, 93–99 (2005).
7. R. P. Agarwal, K. Perera, D. O'Regan, Multiple positive solutions of singular and nonsingular discrete problems via variational methods, Nonl. Anal. TMA, 58, 69–73 (2004).
8. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, Prog. Fract. Differ. Appl., 1, 73–85 (2015).
9. S. Dhar, L. Kong, A critical point approach to multiplicity results for a fractional boundary value problem, Bulletin of the Malaysian Mathematical Sciences Society, 1–17 (2020).
10. W. Dong, J. Xu, D. ORegan, Solutions for a fractional difference boundary value problem, Adv. Difference Equ., 2013, 1–12 (2013)

11. D. Q. Jiang, P. Y. H. Pang, R. P. Agarwal, Upper and lower solutions method and a superlinear singular discrete boundary value problem, Dynam. Systems Appl., 16, 743–754 (2007).
12. C. Goodrich, A. C. Peterson, Discrete Fractional Calculus, Springer international publishing Switzerland, New York, 2015.
13. M. Khaleghi Moghadam, *A survey on multiplicity results for fractional difference equations and variational method, Mathematical Analysis and Convex Optimization,* **3(1)**, 35–58 (2022).
14. M. Khaleghi Moghadam, A survey on existence of a solution to fractional difference boundary value problem with $|u|^{p-2}u$ term, *Mathematical Analysis and Convex Optimization,* **4(1)**, 45–59 (2024).
15. M. Khaleghi Moghadam, M. Avci, Existence results to a nonlinear $p(k)$-Laplacian difference equation, J. Difference Equ. Appl., 23, 1055–1068 (2017).
16. M. Khaleghi Moghadam, J. Henderson, Triple solutions for a dirichlet boundary value problem involving a perturbed discrete $p(k)$- laplacian operator, Open Math. J., 15, 1075–1089 (2017).
17. D. Jiang, J. Chu, D. ORegan, R. P. Agarwal,Positive solutions for continuous and discrete boundary value problems to the one-dimension p-Laplacian, Math. Inequal. Appl., 7, 523–534 (2004).
18. D. Jiang, D. ORegan, R. P. Agarwal, A generalized upper and lower solution method for singular discrete boundary value problems for the one-dimensional p-Laplacian, J. Appl. Anal., 11, 35–47 (2005).
19. D. Jiang, D. ORegan, R. P. Agarwal, Existence theory for single and multiple solutions to singular boundary value problems for the one-dimension p-Laplacian, Adv. Math. Sci. Appl., 13, 179–199 (2003).
20. D. Jiang, L. Zhang, D. ORegan, R. P. Agarwal, Existence theory for single and multiple solutions to singular positone discrete Dirichlet boundary value problems to the one-dimension p-Laplacian, Archivum Mathematicum (Brno), 40, 367–381 (2004).
21. P. Li, L. Xu, H. Wang, Y. Wang, The existence of solutions for perturbed fractional differential equations with impulses via Morse theory, Boundary Value Problems, 2020, 1–13 (2020).
22. K. Matthews, Elementary linear algebra, 1996.
23. H. L. Gray, N. F. Zhang, On a new definition of the fractional difference, Math. Comp., 50, 513–529 (1988).
24. S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional integrals and derivatives: Theory and Applications, Breach Science Publishers: London, UK, 1993.
25. M. E. Samei, G. K. Ranjbar, V. Hedayati, Existence of solutions for equations and inclusions of multi-term fractional q-integro-differential with non-separated and initial boundary conditions, J. Inequal. Appl., 2019, 1–48 (2019).
26. E. Shishkina, S. Sitnik, Transmutations, singular and fractional differential equations with applications to mathematical physics, Academic Press, 2020.
27. M. Struwe, Variational methods, Springer, 2000.

ARTICLE IN PRESS