Global Analysis and Discrete Mathematics
Volume 8, Issue 1, pp. 65–79
ISSN: 2476-5341 *Research Article*

A Numerical Approach based on Differential Quadrature Method for Nonlinear Heat Equation

Taher Rahimi Shamami · Javad Damirchi

Received: 21 May 2024 / Accepted: 14 June 2024

Abstract In this research paper, a numerical method for one- and two- dimensional heat equation with nonlinear diffusion conductivity and source terms is proposed. In this work, the numerical technique is based on the polynomial differential quadrature method for discretization of the spatial domain. The resulting nonlinear system time depending ordinary differential equations is discretize by using the second order Runge-Kutta methods. The Chebyshev-Gauss-Lobatto points in this paper are used for collocation points in spatial discretization. We study accuracy in terms of L_∞ error norm and maximum absolute error along time levels. Finally, several test examples demonstrate the accuracy and efficiency of the proposed schemes. It is shown that the numerical schemes give better solutions. Moreover, the schemes can be easily applied to a wide class of higher dimension nonlinear diffusion equations.

Keywords Polynomial differential quadrature method · Nonlinear heat equations · Runge-Kutta method

Mathematics Subject Classification (2010) 80A20 · 65Z05 · 65L06

1 Introduction

Heat and mass transfer are useful models used to describe heat and mass distribution in various systems. These equations are important in engineering

T. Rahimi Shamami
Department of Mathematics, Faculty of Mathematics, Statistics & Computer Sciences, Semnan, Iran;
E-mail: taherrahimi@semnan.ac.ir

J. Damirchi (Corresponding Author)
Department of Mathematics, Faculty of Mathematics, Statistics & Computer Sciences, Semnan, Iran;
Tel.: +982331535712
Fax: +982331535758
E-mail: damirchi@semnan.ac.ir

and scientific fields such as thermodynamics, fluid mechanics, heat transfer, evaporation, oxygenation, geothermal reservoirs and thermal energy storage.

There are many technical aspects for heat transfer process. The study of heat transfer has attracted much attention. Many publications contain solutions to heat transfer problems can be found in [1]-[17]. In the real world, the problem domain is complex and there is no simple analytical solution for the differential equations arising from the heat transfer process. Some examples of differential equation include the heat transfer equation are diffusion equation for heat transfer analysis, Navier-Stokes equation for fluid dynamics analysis, and the bio-heat transfer equation. Heat transfer processes are often modeled as nonlinear differential equations. In fact, the nonlinear differential equation that governs the temperature field of the system usually occurs when residual energy is supplied to the system. Temperature-dependent characteristics, modeling the temperature dependence of a convective heat transfer coefficient, and other natural process are examples of situations in which nonlinear differential equations arise. Nonlinear differential equations are solved numerically because they do not have closed-form analytical solutions. Verification of numerical solutions is necessary for best practices. The result of mathematical modeling of heat transfer phenomena is often nonlinear differential equations that can be solved using semi-analytical and numerical techniques. An exact analytical solution to the problem of convective fin is given in [1]. In [2] the Adomian decomposition method (ADM) for the analysis of a convective fin with temperature dependence is investigated. The differential transformation method (DTM) is used to solve a similar problem in [3]. In [4], the nonlinear equation describing the temperature distribution is solved using the homotopy analysis method (HAM). Another numerical technique for solving nonlinear differential problems that arise during the heat transfer process is the Green function approach [6]. There are many numerical methods such as, finite element method [7,8], finite difference method [9,10], finite volume method [11,12], lattice Boltzmann method [13,14], and other numerical methods for solving highly nonlinear problems in heat and mass transfer. The differential quadrature (DQ) technique as a numerical approach is employed to solve initial and boundary problems numerically. It was initially developed in the early 1970s by the late Richard Bellman and his colleagues, and it has since been effectively applied to a wide range of engineering and physical scientific challenges. The differential quadrature method (DQM) has been developed and is derived from the interpolation function [18]-[20].

Many researchers have suggested that the differential quadrature method is an accurate technique requires less computational effort. The proposed method has been applied in various computational mechanic's contexts [21]-[27]. Differential quadrature has proven to be an effective technique that replaces the traditional method for solving initial and boundary problems. The method of differential quadrature is based on approximating the derivatives of a function at a sample point as a weighted linear summation of functional values at all sample points in the overall domain of that variable. Using this approximation, the differential equation is then converted into an algebraic equations system.

The differential quadrature weighted coefficients can be obtained using different techniques. To overcome the numerically poor conditions in determining the weighted coefficients, the Lagrangian interpolation polynomial is introduced [28]-[31].

Motivated by the above research, the objective of this study is to investigate the nonlinear heat transfer equation with combined effect of nonlinear heat capacity, thermal conductivity, and source terms, along with initial and boundary conditions in an arbitrary domain in one and two dimensional cases. The governing equation for transfer process is formulated with the help of nonlinear partial differential equations, which are further transformed into a system of time dependent ordinary differential equations using proposed numerical approach. Numerical solutions have been obtained with the help of differential quadrature method and Chebyshev-Gauss- Lobatto collocation points. Numerical simulations of some test problems are presented and discussed.

This structure of the paper is organized as follows; after the introduction in Section 1. Section 2 describes the mathematical formulation of the problem. Section 3 is devoted to introduce the differential quadrature technique for the numerical solution methodology for governing differential equation. In Section 4, the time discretization of the problem for resulting time dependent system of ordinary differential equation with the help of second order Runge–Kutta method (RKM) is introduced. Section 5, illustrates the applying the proposed numerical approach for some test problems and analysis of the results and considering the efficiency of the algorithm and finally, Section 6 is devoted to conclusions.

2 Problem Formulation

In this work, the transient heat conduction equation can be described by the following governing equation in dimensionless form is considered

$$C\left(u\right)\frac{\partial u}{\partial t}=\nabla.\left(K\left(u\right)\nabla u\right)+F\left(u\right),\quad \mathrm{X}\in D\quad t\in(0,T), \tag{1}$$

which can be written in the form

$$C\left(u\right)\frac{\partial u}{\partial t}=K\left(u\right)\nabla^{2}u+\frac{\partial K}{\partial u}\left(\nabla u.\nabla u\right)+F\left(u\right).$$

Without loss of generality and for simplicity, the governing equation (1) can be rewritten in the following simple form, and therefore the following equation is considered

$$\frac{\partial u}{\partial t}=K\left(u\right)\nabla^{2}u+\frac{\partial K}{\partial u}\left(\nabla u.\nabla u\right)+F\left(u\right), \tag{2}$$

with the following initial condition

$$u(\mathrm{X},0)=f\left(\mathrm{X}\right),\quad \mathrm{X}\in D, \tag{3}$$

and the following Dirichlet or Neumann boundary conditions

$$u(\mathrm{X},\mathrm{t}) = p(\mathrm{X},t), \quad \mathrm{X} \in \partial D_1, \tag{4}$$

or

$$\frac{\partial u}{\partial n}(\mathrm{X},\mathrm{t}) = q(\mathrm{X},t), \quad \mathrm{X} \in \partial D_2, \tag{5}$$

where $\partial D = \partial D_1 \cup \partial D_2$ is the boundary of the region $D \subseteq \mathbb{R}^n (n = 1, 2)$. In this problem $u(\mathrm{X},\mathrm{t})$ is the temperature at time t in the space vector X. $C(u)$, $K(u)$, and $F(u)$ are the heat capacity, thermal conductivity and source terms, respectively. In this work, the space vector X is assumed to X=x in one dimensional case and $\mathrm{X} = (x, y)$ in two dimensional case. f, p, and q are known functions of space and time. The aim of this research is to solve the direct nonlinear heat conduction problem to obtain temperature distribution of $u(\mathrm{X},\mathrm{t})$ at every point (X, t). The operator ∇u denote the gradient of u and $\nabla^2 u$ the Laplacian operator of u.

3 Differential Quadrature Method for Space Discretization

The discretization technique is necessary to obtain an appropriate solution of a proposed mathematical problem considered in this paper. Therefore, two main steps of the numerical approach were presented in this work are as follow: In the first step, the space variable is discretized by using the differential quadrature method and, in this way, the original problem is transformed into a system of ordinary differential equations. In the second step, time dependent system of ordinary differential equations is solved by using the RKM.

3.1 One-Dimensional Approximation

For the single variables function $u(x)$, the first and second order derivatives of u at a point x_i are approximated by

$$u(x_i) = \frac{df}{dx}\Big|_{x=x_i} = \sum_{j=1}^{N} a_{ij} u(x_j), \qquad i = 1, 2, \ldots, N, \tag{6}$$

$$u'(x_i) = \frac{df}{dx}\Big|_{x=x_i} = \sum_{j=1}^{N} b_{ij} u(x_j), \qquad i = 1, 2, \ldots, N, \tag{7}$$

$$u''(x_i) = \frac{df}{dx}\Big|_{x=x_i} = \sum_{j=1}^{N} c_{ij} u(x_j), \qquad i = 1, 2, \ldots, N, \tag{8}$$

where a_{ij}, b_{ij}, and c_{ij} are the weighting coefficients and N is the number of grid points in the whole domain. It should be noted that the weighting coefficients vary depending on the location of x_i since they depend on coordinates of the points. The important procedure in DQ approximation is to determine the weighting coefficients a_{ij}, b_{ij}, and c_{ij} efficiently. When the function $u(x)$ is approximated by a high order polynomial, one needs some explicit formulations to compute the weighting coefficients within the scope of a high order polynomial approximation and a linear vector space. Here, for generality, two sets of base polynomials are used to determine the weighting coefficients [31]. In this paper, set of base polynomials is chosen as the Lagrange interpolated polynomials. Define

$$r_k\left(x\right) = \frac{M(x)}{(x-x_k)M^{(1)}(x)}, \quad k = 1, 2, \ldots, N,$$

where

$$M\left(x\right) = \left(x - x_1\right)\left(x - x_2\right)\ldots\left(x - x_N\right),$$

and

$$M^{(1)}\left(x_k\right) = \prod_{j=1, j\neq k}^{N} \left(x_k - x_j\right).$$

So, the coefficients b_{ij} and c_{ij} in the first and second order derivatives of $u(x)$ at the point x_i become

$$r'_k\left(x_i\right) = b_{ik}, \qquad r''_k(x_i) = c_{ik}.$$

We finally obtain the coefficients b_{ij} and c_{ij} after some computations as follows [31]:

$$b_{ij} = \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)}, \qquad i \neq j, \qquad i, j = 1, 2, \ldots, N, \tag{9}$$

$$c_{ij} = 2b_{ij}(b_{ii} - \frac{1}{x_i - x_j}), \qquad i \neq j, \qquad i, j = 1, 2, \ldots, N, \tag{10}$$

$$b_{ii} = -\sum_{j=1, i\neq j}^{N} b_{ij}, \tag{11}$$

$$c_{ii} = -\sum_{j=1, i\neq j}^{N} c_{ij}. \tag{12}$$

3.2 Two Dimensional Approximations

In previous subsection DQM was introduced for one-dimensional case. In two-dimensional case the polynomial differential quadrature (PDQ) is applied for the discretization of space derivatives of the unknown $u(x, y, t)$ [31].

The first space derivatives of u with respect to x and y respectively, can be defined as follows:

$$\frac{\partial u}{\partial x}(x, \mathrm{y}, \mathrm{t})\Big|_{(x=x_i, y=y_j)} = \sum_{k=1}^{N} {w_{ik}}^{(1)} u_{kj}, \tag{13}$$

$$\frac{\partial u}{\partial y}(x,\mathrm{y},\mathrm{t})\Big|_{(x=x_i,y=y_j)}=\sum_{k=1}^{M}\overline{w}_{jk}{}^{(1)}u_{ik}, \tag{14}$$

$$\frac{\partial^2 u}{\partial x^2}(x,\mathrm{y},\mathrm{t})\Big|_{(x=x_i,y=y_j)}=\sum_{k=1}^{N}w_{ik}{}^{(2)}u_{kj} \tag{15}$$

$$\frac{\partial^2 u}{\partial y^2}(x,\mathrm{y},\mathrm{t})\Big|_{(x=x_i,y=y_j)}=\sum_{k=1}^{M}\overline{w}_{jk}{}^{(2)}u_{ik}, \tag{16}$$

where N and M are the number of grid points in the x and y directions, respectively, and $i=1,2,\ldots,N$ and $j=1,2,\ldots,M$.
$w_{ik}{}^{(1)}$ and $w_{ik}{}^{(2)}$ are the DQ weighting coefficients of the first and second derivative of u with respect to x and in a similar manner $\overline{w}_{jk}{}^{(1)}$and $\overline{w}_{jk}{}^{(2)}$ are defined in the y direction. Those coefficients are computed analogous to the coefficients b_{ij} and c_{ij} given in equations (9)-(12) for one dimensional case as follows:

$$w_{ik}{}^{(1)}=\frac{M^{(1)}(x_i)}{(x_i-x_k)M^{(1)}(x_k)},\quad i\neq k, \tag{17}$$

$$\overline{w}_{jk}{}^{(1)}=\frac{M^{(1)}(y_j)}{(y_j-y_k)M^{(1)}(y_k)},\quad j\neq k, \tag{18}$$

$$w_{ik}{}^{(2)}=2w_{ik}{}^{(1)}(w_{ii}{}^{(1)}-\frac{1}{x_i-x_k}),\quad i\neq k,\quad i,k=1,2,\ldots,N, \tag{19}$$

$$\overline{w}_{jk}{}^{(2)}=2\overline{w}_{jk}{}^{(1)}(\overline{w}_{jj}{}^{(1)}-\frac{1}{y_j-y_k}),\quad j\neq k,\quad j,k=1,2,\ldots,M, \tag{20}$$

$$w_{ii}{}^{(1)}=-\sum_{j=1,i\neq j}^{N}w_{ij}{}^{(1)},w_{ii}{}^{(2)}=-\sum_{j=1,i\neq j}^{N}w_{ij}{}^{(2)},\quad i=1,2,\ldots,N, \tag{21}$$

$$\overline{w}_{jj}{}^{(1)}=-\sum_{i=1,i\neq j}^{M}\overline{w}_{ji}{}^{(1)},\overline{w}_{jj}{}^{(2)}=-\sum_{i=1,i\neq j}^{M}\overline{w}_{ji}{}^{(2)},\quad j=1,2,\ldots,M. \tag{22}$$

3.3 Discretization of the Governing Equation

In previous subsections DQ algorithm was introduced for discretization of space coordinates. The governing equation (2) in one and two dimensions, at interior grids of region D can be discretized by using the DQ method as follows:
For one dimensional case, based on the equations (13) and (15) at interior point x_i, $i=2,3,\ldots,N-1$ of domain D, we have

$$\begin{aligned}C\left(u(x_i,\mathrm{t})\right)\frac{\partial u}{\partial t}(x_i,\mathrm{t})=&\,K\left(u\left(x_i,\mathrm{t}\right)\right)\nabla^2u(x_i,\mathrm{t})+\frac{\partial K}{\partial u}\left(x_i,\mathrm{t}\right)\left(\nabla u(x_i,\mathrm{t}).\nabla u(x_i,\mathrm{t})\right)\\&+F(u(x_i,\mathrm{t})),\end{aligned}$$

$$C\left(u(x_i,\mathrm{t})\right)\frac{\partial u}{\partial t}(x_i,\mathrm{t}) = K\left(u\left(x_i,\mathrm{t}\right)\right)\frac{\partial^2 u}{\partial x^2}(x_i,\mathrm{t})+\frac{\partial K}{\partial u}\left(x_i,\mathrm{t}\right)\left(\frac{\partial u}{\partial x}(x_i,\mathrm{t})\right)^2 + F(u(x_i,\mathrm{t})),$$

$$C\left(u(x_i,\mathrm{t})\right)\frac{\partial u}{\partial t}(x_i,\mathrm{t}) = K\left(u\left(x_i,\mathrm{t}\right)\right)\sum_{k=2}^{N-1} w_{ik}{}^{(2)}u(x_k,\mathrm{t}) \quad \frac{\partial K}{\partial u}(x_i,t)\Big(\sum_{k=2}^{N-1} w_{ik}{}^{(1)}u(x_k,t)\Big)^2 + F(u(x_i,\mathrm{t})).$$

By setting

$$u_i = u\left(x_i,\mathrm{t}\right), \qquad \left.\frac{\partial u}{\partial t}\right|_{x_i} = \frac{\partial u}{\partial t}(x_i,\mathrm{t}), \qquad \left.\frac{\partial K}{\partial u}\right|_{x_i} = \frac{\partial K}{\partial u}\left(u(x_i,\mathrm{t})\right),$$
$$C(u_i) = C\left(u(x_i,\mathrm{t})\right), \quad K(u_i) = K\left(u\left(x_i,\mathrm{t}\right)\right), \quad F(u_i) = F(u(x_i,\mathrm{t})),$$

we obtain

$$C\left(u_i\right)\left.\frac{\partial u}{\partial t}\right|_{x_i} = K\left(u_i\right)\sum_{k=2}^{N-1} w_{ik}{}^{(2)}u_k + \left.\frac{\partial K}{\partial u}\right|_{x_i}\left(\sum_{k=2}^{N-1} w_{ik}{}^{(1)}\ u_k\right)^2 + F\left(u_i\right), \tag{23}$$

where $i = 2, 3, \ldots, N-1$. In a similar manner, for two dimensional case at interior points (x_i, y_j) for $i = 2, 3, \ldots, N-1$ and $j = 2, 3 \ldots, M-1$ in domain D, the governing equation (2) is discretized based on DQ method as follows:

$$\begin{aligned}C\left(u(x_i,y_j,\mathrm{t})\right)\frac{\partial u}{\partial t}(x_i,y_j,\mathrm{t}) &= K\left(u\left(x_i,y_j,\mathrm{t}\right)\right)\nabla^2 u(x_i,y_j,\mathrm{t})\\ &+ \frac{\partial K}{\partial u}\left(x_i,y_j,\mathrm{t}\right)\left(\nabla u(x_i,y_j,\mathrm{t}).\nabla u(x_i,y_j,\mathrm{t})\right)\\ &+ F(u(x_i,y_j,\mathrm{t})),\end{aligned}$$

$$\begin{aligned}C\left(u(x_i,y_j,\mathrm{t})\right)\frac{\partial u}{\partial t}(x_i,y_j,\mathrm{t}) &= K\left(u\left(x_i,y_j,\mathrm{t}\right)\right)\left\{\frac{\partial^2 u}{\partial x^2}(x_i,y_j,\mathrm{t})+\frac{\partial^2 u}{\partial y^2}(x_i,y_j,\mathrm{t})\right\}\\ &+ \frac{\partial K}{\partial u}\left(x_i,y_j,\mathrm{t}\right)\left\{\left(\frac{\partial u}{\partial x}(x_i,y_j,\mathrm{t})\right)^2 + \left(\frac{\partial u}{\partial y}(x_i,y_j,\mathrm{t})\right)^2\right\}\\ &+ F(u(x_i,y_j,\mathrm{t})),\end{aligned}$$

and finally we obtain

$$\begin{aligned}C(u_{ij})\left.\frac{\partial u}{\partial t}\right|_{(x_i,y_j)} &= K(u_{ij})\left\{\sum_{k=2}^{N-1} w_{ik}{}^{(2)}u_{kj} + \sum_{k=2}^{M-1}\overline{w}_{jk}{}^{(2)}u_{ik}\right\}\\ &+ \left.\frac{\partial K}{\partial u}\right|_{(x_i,y_j)}\left\{\left(\sum_{k=2}^{N-1} w_{ik}{}^{(1)}\ u_{kj}\right)^2 + \left(\sum_{k=2}^{M-1}\overline{w}_{jk}{}^{(1)}u_{ik}\right)^2\right\}\\ &+ F(u_{ij}),\end{aligned} \tag{24}$$

where $i = 2, 3, \ldots, N-1$ and $j = 2, 3, \ldots, M-1$.

3.4 Implementation of Boundary Conditions

The insertion of Dirichlet type boundary conditions is straightforward since the solution is already known on the boundary. If the boundary condition is Neumann type boundary condition that involves normal derivatives of the unknown function u then these derivatives can also be approximated by DQM. The normal derivative of u can be written as

$$\frac{\partial u}{\partial n}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial n}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial n},$$
$$\frac{\partial u}{\partial n}=\frac{\partial u}{\partial x}x_n+\frac{\partial u}{\partial y}y_n,$$

and $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are discretized by using PDQ method.

$$\frac{\partial u_{ij}}{\partial x}=\sum_{k=1}^{N} w_{ik}{}^{(1)}\ u_{kj}, \qquad i=1,N, \qquad j=1,2,\ldots,M,$$
$$\frac{\partial u_{ij}}{\partial y}=\sum_{k=1}^{N} \overline{w}_{jk}{}^{(1)}\ u_{ik}, \qquad j=1,M, \qquad i=1,2,\ldots,N,$$

and so

$$\left.\frac{\partial u}{\partial n}\right|_{(x_i,y_j,t)}=\left.\frac{\partial u}{\partial x}\right|_{(x_i,y_j,t)}x_n+\left.\frac{\partial u}{\partial y}\right|_{(x_i,y_j,t)}y_n$$
$$=\Big(\sum_{k=1}^{N} w_{ik}{}^{(1)}\ u_{kj}\Big)\ x_n+\Big(\sum_{k=1}^{N} \overline{w}_{jk}{}^{(1)}\ u_{ik}\Big)\ y_n.$$

Thus, equations found by discretizing the normal derivatives of u on the boundary are updated using interior u values which are not known yet. This problem is handled during the iterations performed in the Runge-Kutta method since the initial u values are given.

3.5 Matrix Representation of Discretized Governing Equation

3.5.1 Matrix Representation for One-Dimensional Case

The equation (23) is a set of DQ algebraic equations which can be written in a matrix form

$$\dot{\boldsymbol{U}}=\boldsymbol{K}\boldsymbol{A}_{\boldsymbol{XX}}\boldsymbol{U}+\overline{\boldsymbol{K}}[\boldsymbol{A}_{\boldsymbol{X}}\boldsymbol{U}]^{\boldsymbol{2}}+\boldsymbol{F},$$

where $\boldsymbol{U}=[u_2,u_3,\ldots,u_{N-1}]^T$ is $(N-2)\times 1$ vector for temperature u at grid points x_i, $\dot{\boldsymbol{U}}$ denotes the $(N-2)\times 1$ vector for time derivative of temperature u $(\frac{\partial u}{\partial t})$ at grid points x_i, $\boldsymbol{K}$ denotes a diagonal matrix of size $(N-2)\times(N-2)$ whose diagonal elements are the thermal conductivity $K(u_i)$, for $i=2,3,\ldots,N-1$, $\overline{\boldsymbol{K}}$ is a diagonal matrix of size $(N-2)\times(N-2)$ whose diagonal elements are the thermal conductivity derivatives with respect to u at nodal points in the domain, $\boldsymbol{F} = [\,F(u_2)\,,\,F(u_3)\,,\ldots,F(u_{N-1})]^T$ is $(N-2)\times 1$ vector contains the information of the source term at grid points of domain. $\boldsymbol{A_{XX}} = ({w_{ij}}^{(2)})_{(N-2)\times(N-2)}$ and $\boldsymbol{A_X} = ({w_{ij}}^{(1)})_{(N-2)\times(N-2)}$ are the matrices of size $(N-2)\times(N-2)$ whose elements are the second and first order weighting coefficients respectively, which are defined in equations (17), (19), and (21).

3.5.2 Matrix Representation for Two-Dimensional Case

The equation (24) is a set of DQ algebraic equations which can be written in a matrix form

$$\dot{\boldsymbol{U}} = \boldsymbol{K} \odot \left[\boldsymbol{A_{XX}U} + \boldsymbol{U}\boldsymbol{A_{YY}}^{\boldsymbol{T}}\right] + \overline{\boldsymbol{K}} \odot \left[\boldsymbol{A_X U} + \boldsymbol{U}\boldsymbol{A_Y}^{\boldsymbol{T}}\right]^{\boldsymbol{2}} + \boldsymbol{F},$$

where $\boldsymbol{U}=(u_{ij})$ and $\dot{\boldsymbol{U}} = \left(\frac{\partial u_{ij}}{\partial t}\right)$ denote the $(N-2)\times(M-2)$ matrices for temperature u and its time partial derivative $\frac{\partial u}{\partial t}$ at grid points (x_i, y_j) for $i = 2, 3, \ldots, N-1$ and $j = 2, 3, \ldots, M-1$. $\boldsymbol{K} = (K(u_{ij}))$, $\overline{\boldsymbol{K}} = (K_u(u_{ij}))$ and $\boldsymbol{F} = (F(u_{ij}))$ represent matrices of size $(N-2)\times(M-2)$ whose elements are the values of thermal conductivity, its derivative to with respect to u and the values of source terms at nodal point (x_i, y_j). $\boldsymbol{A_{XX}} = ({w_{ij}}^{(2)})_{(N-2)\times(N-2)}$ and $\boldsymbol{A_X} = ({w_{ij}}^{(1)})_{(N-2)\times(N-2)}$ are the matrices of size $(N-2)\times(N-2)$ whose elements are the second and first order weighting coefficients is x-direction and $\boldsymbol{A_{YY}} = ({\overline{w}_{jk}}^{(2)})_{(M-2)\times(M-2)}$ and $\boldsymbol{A_Y} = ({\overline{w}_{jk}}^{(1)})_{(M-2)\times(M-2)}$ are the matrices of size $(M-2)\times(M-2)$ whose elements are the second and first order weighting coefficients is y-direction. The symbol $\odot$ is the Hadamard product operator which is defined for two matrices A and B of the same dimension $m\times n$ and is denoted by $A\odot B$.

4 Time Discretization

The second order RKM is applied to solve the resulting nonlinear system of time dependent ordinary differential equations obtained by the DQM discretization of the space derivatives. The proposed method uses the n- time level values to find the solution at $(n+1)$-time level. By considering the matrix representation in one-dimensional case as follows:

$$\dot{\mathrm{U}} = \boldsymbol{K A_{XX} U} + \overline{\boldsymbol{K}}[\boldsymbol{A_X U}]^{\boldsymbol{2}} + \boldsymbol{F}.$$

By rewritten the matrix representation in the sample initial value problem $\dot{\mathrm{U}} = f(t, \{\boldsymbol{U}\})$. So,

$$f(t, \{\boldsymbol{U}\}) = \boldsymbol{K A_{XX} U} + \overline{\boldsymbol{K}}[\boldsymbol{A_X U}]^{\boldsymbol{2}} + \boldsymbol{F}.$$

Thus, the second order RKM gives for the governing nonlinear heat equation the following vector equation

$$\{\boldsymbol{U^{n+1}}\} = \{\boldsymbol{U^n}\} + \frac{\Delta t}{2}\left[\{K_1\} + \{K_2\}\right],$$

where

$$\begin{aligned} \{K_1\} &= f(t^n, \{\boldsymbol{U^n}\}), \\ \{K_2\} &= f(t^n + \Delta t, \{\boldsymbol{U^n}{+}\Delta t K_1\}), \end{aligned}$$

where t^n denotes the n-time level and Δt is the time incerement. By using the $t^0 = 0$ time level as the initial condition of problem, the next time levels can be computed. The same approach can be applied in two-dimensional case.

4.1 Choice of Grid Points

Since the weighting coefficients (17)-(22) corresponding to the discretization of the first and second order derivatives in x and y directions respectively, contain grid points x_i, y_j's, the choice of these grid points becomes quite important. Equally spaced grid points, due to their obvious convenience, have been in use by most investigators. However, unequally spaced grid points especially the zeros of orthogonal polynomials like Legendre and Chebyshev polynomials usually give more accurate solutions than the equally spaced grid points.

The so-called Chebyshev-Gauss-Lobatto point distribution offer a better choice and have been found consistently better than the equally spaced, Legendre and Chebyshev points in a variety of problems. For one- dimensional case these grid points on the interval $[a, \mathrm{b}]$ are given as follows:

$$x_i = \frac{a+b}{2} - \frac{a-b}{2}\cos\frac{(i-1)\pi}{N-1}, \quad i = 1, 2, \ldots, N.$$

For the domain $[0, \mathrm{a}] \times [0, \mathrm{b}]$ in two-dimensional case, Chebyshev-Gauss-Lobatto point are defined as follows in the x and y direction, respectively.

$$x_i = \frac{1}{2}\left[1 - \cos\left(\frac{i-1}{N-1}\right)\pi\right]a, \qquad i = 1, 2, \ldots, N,$$

$$y_j = \frac{1}{2}\left[1 - \cos\left(\frac{j-1}{N-1}\right)\pi\right]b, \qquad j = 1, 2, \ldots, M.$$

5 Numerical Simulations

In this section, we will give some numerical examples to illustrate the effectiveness of the proposed algorithm given in the previous sections. We also use the infinity-norm of absolute error *u(x, t)*, to measure the accuracy as follows at the nth time step for one and two dimensional cases, respectively

$$L_\infty = \|u_{Exact} - u_{Approx}\|_\infty = \max_{1\le i\le N} |u_{Exact}(x_i, t_n) - u_{Approx}(x_i, t_n)| \ ,$$

$$L_\infty = \|u_{Exact} - u_{Approx}\|_\infty = \max_{\substack{1\le i\le N\\ 1\le j\le M}} |u_{Exact}(x_i, y_j, t_n) - u_{Approx}(x_i, y_j, t_n)| \ ,$$

where u_{Exact} and u_{Approx} denote the exact and approximate solutions, respectively.

Example 1 In order to show the ability of the numerical method based on the strategy proposed in this work, the problem (1)-(5) in the interval $[-1, 1]$ is considered with the following input data

$$C(u) = 1, \quad K(u) = 1, \quad F(u) = 1 - u,$$

with the initial condition

$$u(x, 0) = 0,$$

and Dirichlet boundary conditions

$$u(-1, \mathrm{t}) = 0, \qquad u(1, \mathrm{t}) = 0.$$

The exact solution for this problem is as follows

$$u(x, t) = 1 - \frac{\cosh x}{\cosh 1} - \frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n \cos((2n-1)\,\frac{\pi}{2}x)}{(2n-1)\left[(2n-1)^2\pi^2 + 4\right]}\ e^{-\left(1+(2n-1)^2\frac{\pi^2}{4}\right)t}.$$

In our computations, several step times are checked for the time interval discretization but finally, for this problem the time increment is chosen with $\Delta t = 0.0001$. The simulation are performed with different number of grid points N in x-direction. The obtained results in terms of the maximum absolute errors at time level $t = 1$ are reported in Table 1. The agreement between the numerical solution and the exact solution can be seen at different number of grid point in x-direction.

Table 1 The maximum absolute errors for Example 1 with $\Delta t = 0.0001$, $t = 1$, and different values of N.

N	30	44	50	100	1000
L_∞	$8.8330e-5$	4.1075e−5	3.1810e−5	7.9538e−6	7.7977e−8

Example 2 In this example, we consider the problem (1) - (5) in the interval [100,200], with input data

$$C(u) = 1, \quad K(u) = \ln u, \quad F(u) = 0,$$

with the initial condition

$$u(x, 0) = e^{-\sqrt{2x}},$$

and Neumann boundary conditions

$$u_x\left(100,t\right) = -\frac{100}{\sqrt{200+2t}}e^{\left(-\sqrt{2x+2t}\right)},$$

$$u_x\left(200,t\right) = -\frac{200}{\sqrt{400+2t}}e^{\left(-\sqrt{2x+2t}\right)}.$$

This problem has the following exact solution

$$u\left(x,t\right) = e^{\left(-\sqrt{2x+2t}\right)}.$$

Similarly to previous example, the number of nodal point is x-direction is taken $N = 5$ for simplicity of computations and the time step for discretization of time interval is selected $\Delta t = 0.01$. In Table 2, The maximum absolute errors at several time levels is reported. The good agreement between the numerical solution and exact solution can be seen.

Table 2 The maximum absolute errors for Example 2 with $\Delta t = 0.01$, and different time levels.

t	0.01	0.1	0.5	1.0	2.0
	1.000e−06	1.000e−06	9.000e−07	7.1000e−07	6.2000e−07

Example 3 In this example the nonlinear heat problem (1)-(5) is considered in two dimensional case in domain $D \times (0,T)$ where

$$D = \{(x,y) \mid 0 < x < 1, \quad 0 < y < 1\},$$

with the following data

$$C\left(u\right) = 1, \quad K\left(u\right) = 1, \quad F\left(u\right) = u(1-u^2).$$

The initial conditions and Dirichlet boundary conditions are taken by appropriate with the analytical solution

$$u(x,y,t) = \frac{1}{2}\tanh\left(\frac{1}{4}x + \frac{1}{4}y + \frac{3}{4}t\right) + \frac{1}{2}.$$

Numerical computations have been performed using the uniform grid in x and y directions and the number of nodal point is x and y direction are taken $N_x = N_y = 80$. In Table 3, The L_∞ error for different time increment for this example is reported. We can define the convergence order for the time accuracy for two time step k_1 and k_2 as follows:

$$\text{Order} = \frac{\log(\frac{L_\infty(k_1)}{L_\infty(k_2)})}{\log(\frac{k_1}{k_2})}.$$

Table 3 The L_∞ error and convergence order with $N = 80$ and $T = 1$ for different time increment.

$k = \Delta t$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{1}{80}$
L_∞	$1.7227e{-}06$	4.3479e−07	1.0865e−07	$2.7163e{-}08$
Order	——	2.0488	2.0133	2.0034

In Table 3, the convergence order for time variable is listed. We can see that the time accuracy of the method for this example is second order, which is consistent with the theoretical results of RKM.

From obtained results for these examples, it can be seen that good accuracy for one and two dimensional cases between exact and numerical solution can be obtained with small number of grid points in space discretization and low computational cost.

6 Conclusions

In this paper, a combination of the DQM method and the RKM are used to solved numerically the heat equations in one- and two- dimensional cases. Heat equations generally are not easy to solve due to nonlinearity form. To solve the problem considered in this paper, DQM is used to discretize spatial derivatives as a method that provides a small number of grid points and easy to implemented with other discretization methods in the domain. With the DQM discretization, the resulting nonlinear system of time dependent ordinary differential equations is solved using the second order RKM. The main idea of this research is to present a coupling method of DQM- space discretization and RKM-time discretization for solving nonlinear heat transfer problem. Unlike the DQM space-time discretization approach, which requires more than one grid point for each time subdomain, leading to a nonlinear system of high-dimensional equations, in the proposed method, the RKM can improve the solution in the time direction by successively solving the system of equations, for each time level, respectively.

References

1. S. Abbasbandy, E. Shivanian, Exact analytical solution of a nonlinear equation arising in heat transfer, Physics Letters A, 374(4), 567–574 (2010).
2. C. Arslanturk, A decomposition method for fin efficiency of convective straight fins with temperature-dependent thermal conductivity, International communications in heat and mass transfer, 32(6), 831–841 (2005).
3. A. Moradi, H. Ahmadikia, Analytical solution for different profiles of fin with temperature-dependent thermal conductivity, Mathematical Problems in Engineering, 2010, (2010).
4. A. Aziz, F. Khani, Convection–radiation from a continuously moving fin of variable thermal conductivity, Journal of the Franklin Institute, 348(4), 640–651 (2011).
5. D. D. Ganji, Z. G. Ziabkhsh, D. H. Ganji, Determination of temperature distribution for annular fins with temperature dependent thermal conductivity by HPM, Thermal science, 15(1), 111–115 (2011).

6. M. R. Jones, P. S. Vladimir, Green's Function Approach to Nonlinear Conduction and Surface Radiation Problems, Heat Transfer Summer Conference, 43567, 45–51 (2009).
7. S. S. M. Hosseini, J. R. Howell, S. H. Mansouri, Inverse boundary design conduction-radiation problem in irregular two-dimensional domains, Numerical Heat Transfer: Part B: Fundamentals, 44(3), 209–224 (2003).
8. L. M. Ruan, M. Xie, H. Qi, W. An, H. P. Tan, Development of a finite element model for coupled radiative and conductive heat transfer in participating media, Journal of Quantitative Spectroscopy and Radiative Transfer, 102(2), 190–202 (2006).
9. F. Asllanaj, A. Milandri, G. Jeandel, J. R. Roche, A finite difference solution of non-linear systems of radiative–conductive heat transfer equations, International Journal for Numerical Methods in Engineering, 54(11), 1649–1668 (2002).
10. B. Safavisohi, E. Sharbati, C. Aghanajafi, S. R. K. Firoozabadi, Finite difference solution for radiative–conductive heat transfer of a semitransparent polycarbonate layer, Journal of applied polymer science, 112(6), 3313–3321 (2009).
11. S. Patankar, Numerical heat transfer and fluid flow, Taylor & Francis, 2018.
12. K. Birgelis, R. Uldis, Convergence of the finite volume method for a conductive-radiative heat transfer problem, Mathematical Modelling and Analysis, 18(2), 274–288 (2013).
13. S. C. Mishra, K. R. Hillol, Solving transient conduction and radiation heat transfer problems using the lattice Boltzmann method and the finite volume method, Journal of Computational Physics, 223(1), 89–107 (2007).
14. B. Mondal, C. M. Subhash, Lattice Boltzmann method applied to the solution of the energy equations of the transient conduction and radiation problems on non-uniform lattices, International Journal of Heat and Mass Transfer, 51(1-2), 68–82 (2008).
15. J. V. Beck, B. Blackwell, C. R. S. Clair, Inverse Heat Conduction-Ill Posed Problem, First ed., Wiley, 1985.
16. C. Y. Yang, Direct and Inverse Solutions of Hyperbolic Heat Conduction Problems, Journal of Thermophysics and Heat Transfer, 19, 217–225 (2005).
17. Y. Gong, Q. Fei, D. Chunying T. N. Jon An isogeometric boundary element method for heat transfer problems of multiscale structures in electronic packaging with arbitrary heat sources, Applied Mathematical Modelling, 109, 161–185 (2022).
18. R. Bellman, J. Casti, Differential Quadrature and Long-Term Integration, Journal of Mathematical Analysis and Application, 34, 235–238 (1971).
19. R. E. Bellman, B. G. Kashef, J. Casti, Differential Quadrature: A Technique for Rapid Solution of Nonlinear Partial Differential Equations, Journal Computational Physics, 10, 40–52 (1972).
20. C. W. Bert, M. Mailk, Differential Quadrature Method in Computational Mechanics: AReview, Applied Mechanics Review, 49, 1–28 (1996).
21. C. W. Bert, S. K. Jang, A. G. Striz, Two New Approximate Methods for Analyzing Free Vibration of Structural Components, AIAA Journal, 26, 612–618 (1988).
22. C. W. Bert, M. Malik, Free Vibration Analysis of Tapered Rectangular Plates by Differential Quadrature Method: A Semi-Analytical Approach, Journal of Sound and Vibration, 190, 41–63 (1996).
23. C. W. Bert, M. Malik, On the Relative Effects of Transverse Shear Deformation and Rotary Inertia on the Free Vibration of Symmetric Cross-Ply Laminated Plates, Journal of Sound and Vibration, 193, 927–933 (1996).
24. C. W. Bert, X. Wang, A. G. Striz, Differential Quadrature for Static and Free Vibration Analysis of Anisotropic Plates, International Journal of Solids and Structures, 30, 1737–1744 (1993).
25. C. W. Bert, X. Wang, A. G. Striz, Convergence of the DQ Method in the Analysis of Anisotropic Plates, Journal of Sound and Vibration, 170, 140–144 (1994).
26. A. G. Striz, W. Chen, C. W. Bert, Static Analysis of Structures by the Quadrature Element Method (QEM), International Journal of Solids and Structures, 31, 2807–2818 (1994).
27. K. M. Liew, J. B. Han, Z. M. Xiao, H. Du, Differential Quadrature Method for Mindlin Plates on Winkler Foundations, International Journal of Mechanical Sciences, 38, 405–421 (1996).
28. J. B. Han, K. M. Liew, Axisymmetric Free Vibration of Thick Annular Plates, International Journal of Mechanical Science, 41, 1089–1109 (1999).

29. K. M. Liew, J. B. Han, Z. M. Xiao, Differential Quadrature Method for Thick Symmetric Cross-Ply laminates with first-order shear flexibility, International Journal of Solids and Structures, 33(18), 2647–2658 (1996).
30. H. Du, K. M. Liew, M. K. Lim, Generalized Differential Quadrature Method for Buckling Analysis, Journal of Engineering Mechanics, 122, 95–100 (1996).
31. C. Shu, Differential Quadrature and its Application in Engineering, Springer Science & Business Media, 2000.