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A Survey on Hamiltonian Cycle in Cayley Graphs

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Abstract It has been conjectured there is a Hamiltonian cycle in every Cayley graph. Interest in this and other closely related questions has grown in the past few years. There have been many papers on the topic, but it is still an open question whether every connected Cayley graph has a Hamiltonian cycle. In this paper, we survey the results, techniques, and open problems in the field.

Keywords Hamiltonian cycle · Cayley graph

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1 Introduction

The general problem of finding the existence of a Hamiltonian cycle is, the well-known Hamiltonicity problem. The problem of finding Hamiltonian cycles in graphs can be studied in all kinds of graphs, and this survey deals with the Lov´asz conjecture about the existence of Hamiltonian paths and cycles in the special case of vertex-transitive graphs called Cayley graphs.

Definition 1 Let S be a non empty subset of a finite group G without the identity element such that $S = S^{-1}$. The Cayley graph $Cay(G, S)$ is the graph whose vertices are the elements of G, with an edge joining g and gs, for every $g \in G$ and $s \in S$.

The Hamiltonian cycle problem in Cayley graphs involves finding a cycle that visits all the vertices of the graph exactly once, and returns to the starting vertex.

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Experimental studies have shown that a Hamiltonian cycle in Cayley graphs depends on various factors such as the size and structure of the underlying group, the choice of generating set, and the properties of the graph itself. For example, it has been observed that Cayley graphs of abelian groups always have Hamiltonian cycles, whereas for non-abelian groups, this may not always be the case. Experimental results have also shown that the degree of the vertices in the Cayley graph is an important parameter in the existence of a Hamiltonian cycle. For instance, in Cayley graphs with high degrees, it is less likely to find a Hamiltonian cycle as the graph becomes disconnected or has many isolated vertices. Furthermore, studies have also considered the connectivity of the Cayley graph as an important factor for the Hamiltonian cycle problem. It has been shown that if a Cayley graph is connected, then the existence of a Hamiltonian cycle is guaranteed. However, when the graph is not connected, the presence of a Hamiltonian cycle becomes more uncertain. In conclusion, a Hamiltonian cycle in Cayley graphs is greatly influenced by various factors such as group structure, generating set size, vertex degree, and graph connectivity. While there are some cases where a Hamiltonian cycle is guaranteed to exist, it remains a challenging problem in general and further research is needed to fully understand and solve it. Chapter two will explore in depth the research findings of different authors within this field.

Interested in this and other closely related questions has grown in the past few years and there have been many papers of Hamiltonian cycles in Cayley graphs. Since the general case is so hard, it is natural to look at special cases. All graphs in this paper are undirected and connected. Most of the results about the conjecture on Cayley graphs were first surveyed in 1984 by Witte and Gallian [10]. It is important to determine which groups with which generating sets have Hamiltonian paths and cycles. Since, this area is so vast, in this survey the main results on Cayley graphs of important groups will be covered. Therefore, the main objective of the survey is to provide a better picture of the famous conjecture on Hamiltonian paths and cycles in Cayley graphs, as it exists today. Using a computer, you can obtain the Hamiltonian cycles of Cayley graphs, which are examined in Section 3.

2 Related Concept and Technique

By Definition 1, vertices of the Cayley graph are the elements of group G. Therefore in order to check the Hamiltonianity of the Cayley graph, the first step is to check the properties of the group under the Cayley graph. In this section, abelian, non-abelian, dihedral, solvable and nilpotent groups and groups with special order have been investigated. Throughout the paper, we have used standard terminology of graph theory and group theory that can be found in [2].

Lemma 1 *[1] Assume G is an* ***abelian group****. Then every connected Cayley graph on G has a hamiltonian cycle.*

Example 1 $Cay(C_2, a)$ is a Cayley graph with two vertices, where $C_2 =< a >$ is cyclic group of order 2. We consider (a, a) as its Hamiltonian cycle which is:

$$e \xrightarrow{a} a \xrightarrow{a} a^2 = e.$$

We will always let $G' = [G; G]$ be the commutator subgroup of G. Then, G/G' is always abelian, so Lemma 1 provides a Hamiltonian cycle in $Cay(G, S)$. For $S \subseteq G$, a sequence $(s_1, s_2, \ldots, s_n)$ of elements of $S \cup S^{-1}$ specifies the walk in the Cayley graph $Cay(G, S)$ that visits the vertices:

$$e, s_1, s_1 s_2, \ldots, s_1, s_1 s_2 \ldots s_n.$$

Also, $(s_1, s_2, \ldots, s_n)^{-1} = (s_1^{-1}, s_2^{-1}, \ldots, s_n^{-1})$.

Definition 2 [3] For any hamiltonian cycle $C = (s_1, s_2, \ldots, s_n)$ in the Cayley graph $Cay(G, S)$, we let

$$V(C) = \prod_{i=1}^{n} s_i = s_1 s_2 \ldots s_n,$$

is a **voltage** of C.

2.1 Groups of dihedral type

Notation. We use D_{2n} and Q_{4n} to denote the dihedral group of order $2n$ and the generalized quaternion group of order $4n$, respectively. That is,

$$D_{2n} = \langle f, x | f^2 = x^n = e, fxf = x^{-1} \rangle,$$
$$Q_{4n} = \langle f, x | x^{2n} = e, f^2 = x^n, f^{-1}xf = x^{-1} \rangle.$$

Definition 3
- A group G is of dihedral type if it has
  - an abelian subgroup A of index 2, and
  - an element f of order 2 (with $f \notin A$),

  such that f inverts every element of A (i.e., $f^{-1}af = a^{-1}$ for all $a \in A$).
- A group G is of quaternion type if it has
  - an abelian subgroup A of index 2, and
  - an element f of order 4, such that f inverts every element of A.

  Thus, dihedral groups are the groups of dihedral type in which A is cyclic, while generalized quaternion groups are the groups of quaternion type in which A is cyclic.

It is not very difficult to show that Cayley graphs on dihedral groups of small order are hamiltonian.

Lemma 2 *[7] If n has at most three distinct prime factors, then every connected Cayley graph on D_{2n} has a hamiltonian cycle.*

A similar argument also yields a result for other groups of dihedral type.

Lemma 3 *If $G = A : Z_2$ is of dihedral type, and $|A|$ is the product of at most three primes (not necessarily distinct), then every connected Cayley graph on G has a hamiltonian cycle.*

The following Corollary is a result of perevious Lemma.

Corollary 1 *If n is the product of atmost three primes (not necessarily distinct), then every connected Cayley graph on any group of quaternion type of order $4n$ has a hamiltonian cycle.*

Remark. If G is a group of dihedral type and $|G|$ is divisible by 4, then, Alspach et al. [11], have shown that every connected Cayley graph on G has a hamiltonian cycle. In fact, the Cayley graphs are hamiltonian connected (or hamiltonian laceable when they are bipartite).

Lemma 4 *If $G = D_{2pq} \times C_r$ (C_r is cyclic group of order r). where p, q, and r are distinct odd primes, then every connected Cayley graph on G has a Hamiltonian cycle.*

According to [5], every abelian group is nilpotent , and also every nilpotent group is solvable. Therefore after the investigation of the Hamiltonicity of Cayley graphs on the abelian group, must be investigated Hamiltonicity of Cayley graphs on nilpotent and solvable groups.

The question of whether connected Cayley graphs on nilpotent groups consistently possess Hamiltonian paths remains unresolved. Previously, it was established that if a nilpotent group G has a prime order $|G| = p^n$, then a Hamiltonian cycle exists in the corresponding Cayley graph $Cay(G, S)$. Subsequent research revealed that if the commutator subgroup $[G, G]$ of a finite group G is cyclic and of prime-power order, then every connected Cayley graph on G contains a Hamiltonian cycle. Attempts have been made to extend this result by considering only the cyclic nature of the commutator subgroup, irrespective of its order, but this has proven to be exceptionally challenging. In 2013, Ghaderpour and Morris tackled this problem by adding the condition on the order of $[G, G]$ and instead requiring G to be nilpotent, yielding new insights.

Theorem 1 *[1]*

(i) *If the commutator subgroup of G has order $2p$, where p is an odd prime, then every connected Cayley graph on G has a hamiltonian cycle.*

(ii) *Let G be a nontrivial, finite group. If G is nilpotent and the commutator subgroup of G is cyclic, then every connected Cayley graph on G has a hamiltonian cycle.*

Morris's work in [8] demonstrated the existence of infinitely many groups G where every Cayley graph on G exhibits a Hamiltonian cycle, despite G not being solvable. Additionally, he presented several infinite families of finite groups G for which every connected Cayley graph on G possesses a Hamiltonian cycle. However, it appears that the combined set of these families comprises only a finite number of groups that are not solvable. In [6], it is asserted that every

Cayley graph constructed on the Alternating group of degree 5 (denoted as A_5), which is recognized as the smallest non-solvable group, contains a Hamiltonian cycle. Hence, there are infinitely many primes p, such that every Cayley graph on $A_5 \times Z_p$ has a Hamiltonian cycle. Specifically, Morris has proven the following result when $p \equiv 1(\text{mod } 30)$.

Theorem 2 *[8] If p is a prime, such that $p \equiv 1(mod\, 30)$, then every connected Cayley graph on the direct product $A_5 \times Z_p$ has a Hamiltonian cycle.*

In the continuation of this section, the Hamiltonicity of the graphs in special order is examined.
The following theorem plays a central role.

Theorem 3 *(Factor Group Lemma [10]) Suppose:*

- *N is a cyclic normal subgroup of G,*
- *$\overline{G} = G/N$ and $C_1 = (\overline{s}_1, \overline{s}_2, \ldots, \overline{s}_n)$ is a hamiltony cycle in $Cay(\overline{G} = G/N, \overline{S})$,*
- *The $V(C) = \overline{s}_1\overline{s}_2 \ldots \overline{s}_n$ generates N,*

then, there is a Hamiltonian cycle in $Cay(G; S)$.

Corollary 2 *Suppose:*

- *S is a generating set of G,*
- *N is a normal subgroup of G, such that $|N|$ is prime,*
- *$sN = tN$ for some $s, t \in S$ with $s \neq t$, and*
- *There is a Hamiltonian cycle in $Cay(G/N; \overline{S})$ that uses at least one edge labeled $\overline{s}$,*

then, there is a Hamiltonian cycle in $Cay(G; S)$.

Proof Let $C = \{\overline{s_1}, \overline{s_2}, \ldots, \overline{s_n}\}$ be a Hamiltonian cycle in $Cay(G/N; S)$, such that $s_i = s$ for some i, and assume, for simplicity, that $i = n$. If $V(C) \neq e$, then since, $|N|$ is a prime number, the subgroup generated by $V(C)$ is N. Thus, Factor Group Lemma 3 applies.
Now, if $V(C) = e$, then let $C = \{\overline{s_1}, \overline{s_2}, \ldots, \overline{s_{n-1}}, \overline{t}\}$. Since,

$$\overline{t} = tN = sN = s_nN,$$

this is another representation of the Hamiltonian cycle C. However,

$$V(C_1) = s_1s_2 \ldots s_{n-1}t = s_1s_2 \ldots s_{n-1}s_n.(s_n)^{-1}t = e(s^{-1}t) \neq e,$$

since, $s \neq t$. So, Factor Group Lemma 3 applies.

The following lemmas and propositions often makes it possible to use Factor Group Lemma 3 for finding Hamiltonian cycles in connected Cayley graphs of G.

Theorem 4 *[1] If the commutator subgroup G' of G is a cyclic p-group, then every connected Cayley graph on G has a Hamiltonian cycle.*

Kutnar et al. in [6] has shown the following theorem for Cayley graphs on groups whose order has few prime factors.

Theorem 5 *[6] Let S be a generating set of a finite group G. If $|G|$ has any of the following forms (where p, q, r and s are distinct primes, and k is a positive integer), then $Cay(G, S)$ has a hamiltonian cycle:*

(1) kp, where $k \leq 47$ ($kp \neq 2$),
(2) kpq, where $k \leq 7$,
(3) pqr,
(4) $pqrs$, where p, q, r, and s are odd prime numbers,
(5) kp^2, where $k \leq 4$,
(6) kp^3, where $k \leq 2$,
(7) p^k.

3 Result and discution

We mention some of the approaches that have been taken:

- Cayley graphs on groups that are almost abelian: commutator subgroup of prime order (or cyclic of prime-power order), commutator subgroup that is cyclic of order pq (where p and q are prime) [7], dihedral groups and nilpotent groups [6].
- Existence of small-valency Cayley graphs that have a Hamiltonian cycle [10].
- Hamiltonian paths (or cycles) in certain Cayley graphs on symmetric groups: These provide a list of all the permutations of a set. Several examples are described in [9].
- Hamiltonian cycles in vertex-transitive graphs (graphs such that all vertices are in the same orbit of the automorphism group): See the survey [6]. Cayley graphs are examples of vertex-transitive graphs.

As mentioned in the introduction, the paper relies heavily on the computer algebra system: standard installation of the computer algebra system GAP and G. Helsgaun's implementation LKH of the Lin-Kernighan heuristic [4]. G. Helsgaun's [4] implementation LKH of the Lin-Kernighan heuristic is a very powerful tool for finding hamiltonian cycles, and the function LKH (X, AdditionalEdges, RequiredEdges) interfaces GAP with this program (It is defined in the file LKH.gap). Given a graph X (in grape format), and two lists of edges, the function constructs a graph X^+ by adding the edges in **AdditionalEdges** to X, and asks LKH to find a hamiltonian cycle in X^+ that contains all of the edges in RequiredEdges. If $X = Cay(G; S)$, then the hamiltonian cycle is returned as a list of elements of G, in the order that they are visited by the cycle.

4 Conclusion

In this survey, the following problem have been studied with the purpose of updating the results on the existence of Hamiltonian paths and cycles in Cayley graphs. Cayley graph on particular group with which generating set has Hamiltonian cycles. It delves into all documented findings from 1995 onwards regarding Hamiltonicity, specifically focusing on Cayley graphs involving dihedral groups, permutation groups, p-groups, nilpotent and solvable groups. Despite this comprehensive exploration, numerous unresolved challenges persist, highlighting the ongoing complexity of the issue. On the whole, the conjecture has been true for groups of small order as well as $Cay(G, S)$ has a Hamiltonian cycle in the cases when G is abelian, when $[G, G]$ is cyclic of prime-power order, when G is of prime-power order, when the order of G is multiples of primes and when G is nilpotent and $[G, G]$ is cyclic.
In fact, the proof of the following theorem is an open problem.

Theorem 6 *Let S be a generating set of a finite group G. If $|G|$ has any of the following forms (where p, q, r and s are distinct primes, and k is a positive integer), then, $Cay(G, S)$ has a hamiltonian cycle:*

(1) kp, where $k \geq 47$ $(kp \neq 2)$,
(2) kpq, where $k \geq 7$.

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