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Unbounded Order-to-Order Continuous Operators on Riesz Spaces

Kazem Haghnejad Azar · Mina Matin · Sajjad Ghanizadeh Zare

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Abstract Let E and F be two Riesz spaces. An operator $T: E \to F$ between two Riesz spaces is said to be unbounded order-to-order continuous whenever $x_\alpha \xrightarrow{uo} 0$ in E implies $Tx_\alpha \xrightarrow{o} 0$ in F for each net $(x_\alpha) \subseteq E$. This paper aims to investigate several properties of a novel class of operators and their connections to established operator classifications. Furthermore, we introduce a new class of operators, which we refer to as order-to-unbounded order continuous operators. An operator $T: E \to F$ between two Riesz spaces is said to be order-to-unbounded order continuous (for short, *ouo*-continuous), if $x_\alpha \xrightarrow{o} 0$ in E implies $Tx_\alpha \xrightarrow{uo} 0$ in F for each net $(x_\alpha) \subseteq E$. In this manuscript, we investigate the lattice properties of a certain class of objects and demonstrate that, under certain conditions, order continuity is equivalent to unbounded order-to-order continuity of operators on Riesz spaces. Additionally, we establish that the set of all unbounded order-to-order continuous linear functionals on a Riesz space E forms a band of $E^\sim$.

Keywords Riesz space · Order convergence · Unbounded order convergence

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K. Haghnejad Azar (Corresponding Author)
Department of Mathematics and Application Faculty of Sciences University of Mohaghegh Ardabili, Ardabil, Iran;
E-mail: haghnejad@uma.ac.ir

M. Matin
Department of Mathematics and Application Faculty of Sciences University of Mohaghegh Ardabili, Ardabil, Iran;
E-mail: minamatin1368@yahoo.com

S. Ghanizadeh Zare
Department of Mathematics and Application Faculty of Sciences University of Mohaghegh Ardabili, Ardabil, Iran;
E-mail: s.ghanizadeh@uma.ac.ir

1 Introduction

The notion of unbounded order convergence, also known as *uo*-convergence, was initially introduced in [5] and further developed in [11]. In recent years, this concept has received significant attention and has been the subject of investigation in several papers, including [4, 6, 7]. An area of particular interest is the study of geometric properties of Banach lattices using uo-convergence. Wickstead provided a characterization of spaces in which weak convergence of nets is equivalent to *uo*-convergence, see [13]. It was followed in [6], Gao characterized the space E such that in its dual space E^*, *uo*-convergence implies w^*-convergence and vice versa. He also characterized the spaces in whose dual space simultaneous *uo*- and w^*-convergence imply weak/norm convergence. Bahramnezhad and Haghnejad Azar have introduced unbounded order continuous operators on Riesz spaces and investigated on the lattices properties of this classification of operators, see [3]. Also in another article, Hanghnejad Azar, Jalili, and Moghimi introduced a new classification of operators as order-to-norm topology continuous operators and order-to-weak topology continuous operators in [9]. They investigated the properties of these operators, and left as an open problem whether every order-to-norm continuous operator from a Riesz space to a normed Riesz space has a modulus. This manuscript introduces a new classification of operators, namely strongly order continuous operators, and investigates their lattice properties. Specifically, we demonstrate that if an order bounded linear functional f on a Riesz space E is strongly order continuous, then its modulus exists and is also strongly order continuous.

Recall that a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in a Riesz space E is *order convergent* (or, *o*-convergent for short) to $x \in E$, denoted by $x_\alpha \xrightarrow{o} x$ whenever there exists another net $(y_\beta)_{\beta\in\mathcal{B}}$ in E such that $y_\beta \downarrow 0$ and for every $\beta \in \mathcal{B}$, there exists $\alpha_0 \in \mathcal{A}$ such that $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_0$. A net (x_α) in a Riesz space E is *unbounded order convergent* (or, *uo*-convergent for short) to $x \in E$ if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for all $u \in E^+$. We denote this convergence by $x_\alpha \xrightarrow{uo} x$ and write that (x_α) *uo*-convergent to x. This is an analogue of pointwise convergence in function spaces. Let $\mathbb{R}^A$ be the Riesz space of all real-valued functions on a non-empty set A, equipped with the pointwise order. It is easily seen that a net (x_α) in $\mathbb{R}^A$ *uo*-converges to $x \in \mathbb{R}^A$ if and only if it converges pointwise to x. For instance in c_0 and $\ell_p(1 \leq p \leq \infty)$, *uo*-convergence of nets is the same as coordinate-wise convergence. Assume that (Ω, Σ, μ) is a measure space and let $E = L_p(\mu)$ for some $1 \leq p < \infty$. Then *uo*-convergence of sequences in $L_p(\mu)$ is the same as almost everywhere convergence. Note that the *uo*-convergence in a Riesz space E does not necessarily correspond to a topology on E. For example, let $E = c$, the Banach lattice of real valued convergent sequences. Put $x_n = \Sigma_{k=1}^n e_k$, where (e_n) is the standard basis. Then (x_n) is *uo*-convergent to $x = (1, 1, 1, ...)$, but it is not norm convergent.

We show that the collection of all order bounded strongly order continuous linear functionals on a Riesz space E is a band of $E^\sim$ where $E^\sim$ is order dual of E [Theorem 2.7]. For unexplained terminology and facts on Banach lattices and positive operators, we refer the reader to [1,2]. Let us start with the definition. Recall that an operator $T\colon E \to F$ between two Riesz spaces is said to be order continuous (resp. σ-order continuous) if $x_\alpha \xrightarrow{o} 0$ (resp. $x_n \xrightarrow{o} 0$) in E implies $Tx_\alpha \xrightarrow{o} 0$ (resp. $Tx_n \xrightarrow{o} 0$) in F. The collection of all order continuous operators of $L_b(E, F)$ (the vector space of all order bounded operators from E to F) will be denoted by $L_n(E, F)$, that is

$$L_n(E, F) := \{T \in L_b(E, F) : \text{T is order continuous}\}.$$

Similarly, $L_c(E, F)$ will denote the collection of all order bounded operators from E to F that are σ-order continuous. That is,

$$L_c(E, F) := \{T \in L_b(E, F) : \text{T is } \sigma\text{-order continuous}\}.$$

Let E, F be two Riesz spaces. Recall from [3], an operator $T\colon E \to F$ between two Riesz spaces is said to be *uo*-continuous, if $x_\alpha \xrightarrow{uo} 0$ in E implies $T(x_\alpha) \xrightarrow{uo} 0$ in F. The collection of all *uo*-continuous operators will be denoted by $L_{uo}(E, F)$. Recall that from [8], a continuous operator $T\colon E \to F$ between two normed Riesz spaces is said to be σ-*uon*-continuous, if for each norm bounded *uo*-null sequence $(x_n) \subseteq E$ implies $T(x_n) \xrightarrow{\|\cdot\|} 0$ in F. When $T\colon E \to F$ is an order bounded, its order adjoint $T'\colon F^\sim \to E^\sim$ satisfies

$$T'(f(x)) = f(T(x)),$$

for all $f \in F^\sim$ and $x \in E$. A Riesz space is said to be laterally complete (resp. σ*-laterally complete*) whenever every subset of pairwise disjoint positive vectors (if every disjoint sequence) has a supremum. For a set A, $\mathbb{R}^A$ is an example of σ-laterally complete Riesz space. A positive non-zero vector a in a Riesz space E is an atom if the ideal I_a generated by a coincides with span a. We say that E is non-atomic if it has no atoms. We say that E is atomic if E is the band generated by all the atoms in it.

Consider an order bounded operator $T\colon E \to F$ between two Riesz spaces with F Dedekind complete. Then the null ideal N_T of T is defined by $N_T = \{x \in E : |T|(|x|) = 0\}$.

2 Unbounded Order-to-order Continuous Operators

Definition 1 An operator $T\colon E \to F$ between two Riesz spaces is said to be:

i. unbounded order-to-order continuous or strongly order continuous (*so*-continuous for short), if $x_\alpha \xrightarrow{uo} 0$ in E implies $Tx_\alpha \xrightarrow{o} 0$ in F for each net $(x_\alpha) \subseteq E$.
ii. σ-unbounded order-to-order continuous or σ-strongly order continuous (σ-*so*-continuous for short), if $x_n \xrightarrow{uo} 0$ in E implies $Tx_n \xrightarrow{o} 0$ in F for each sequence $(x_n) \subseteq E$.

The collection of all *so*-continuous operators of $L_b(E,F)$ will be denoted by $L_{so}(E,F)$, that is

$$L_{so}(E,F) := \{T \in L_b(E,F) : \text{T is } so\text{-continuous}\}.$$

Similarly, $L_{\sigma-so}(E,F)$ will denote the collection of all order bounded operators from E to F that are σ-*so*-continuous. That is,

$$L_{\sigma-so}(E,F) := \{T \in L_b(E,F) : \text{T is } \sigma\text{-}so\text{-continuous}\}.$$

Example 1 Let E be a Riesz space, $e \in E^+$ and B_e be a band generated by e in E. The operator $T\colon E \to B_e$ that defined by $T(x) = |x| \wedge e$ is a *so*-continuous operator.

Remark 1 1. The class of *so*-continuous operators differ from the calss of *uo*-continuous operators. For example the identity operator $I\colon c_0 \to c_0$ is a *uo*-continuous operator, while it is not *so*-continuous.
2. Let F has order continuous norm. If $T\colon E \to F$ is *so*-continuous, then it is a weakly compact operator. Let $(x_n) \subseteq E$ be norm bounded and *uo*-null sequence. By assumption $T(x_n) \xrightarrow{o} 0$ in F. Because F has order continuous norm, $(T(x_n))$ is norm-null in F. So T is a σ-*uon*-continuous operator. By Remark 2.9 of [8], T is M-weakly compact and therefore is weakly compact.

If $T\colon E \to F$ is a *so*-continuous operator, then it is also order continuous. In the following example we show that the converse is not true in general.

Example 2 The identity $I\colon \ell^1 \to \ell^1$ is order continuous, while it is not *so*-continuous. Because $(e_n) \subseteq \ell^1$ is *uo*-null while it is not *o*-null.

As we said, if $T\colon E \to F$ is a *so*-continuous operator, it is an order continuous and so it is an order bounded operator. In the following example we show that the converse is not true in general.

Example 3 1. The identity operator $I\colon c_0 \to c_0$ is order continuous and therefore is order bounded but is not *so*-continuous. Indeed, the standard basis sequence of c_0 is *uo*-converges to 0 but is not order convergent.
2. The operator $T\colon \ell^1 \to \ell^\infty$ defined by

$$T(x_1, x_2, \ldots) = (\sum_{i=1}^{\infty} x_i, \sum_{i=1}^{\infty} x_i, \ldots),$$

is order bounded. Now if $(e_n)_n$ is the standard basis of ℓ^1, then $e_n \xrightarrow{uo} 0$ in ℓ^1 and $T(e_n) = (1, 1, 1, \ldots)$. Therefore T is not *so*-continuous.

Proposition 1 *1. Let E, F be two Riesz spaces such that E is finite-dimensional. Then* $L_{so}(E,F) = L_n(E,F)$ *and* $L_{\sigma-so}(E,F) = L_c(E,F)$.
2. Let E, F be two Riesz spaces such that F is finite-dimensional. Then $L_{so}(E,F) = L_{uo}(E,F)$ *and* $L_{\sigma-so}(E,F) = L_{\sigma-uo}(E,F)$.
3. Let G be a sublattice of E. If $T \in L_{so}(E,F)$*, then* $T \in L_{so}(G,F)$.

Proof 1. (and 2.) Follows immediately if we observe that in a finite-dimensional Riesz space order convergence is equivalent to *uo*-convergence.

3. Let $(x_\alpha) \subseteq G$ be a *uo*-null net. It is obvious that (x_α) is *uo*-null in E. By assumption, we have $T(x_\alpha) \xrightarrow{o} 0$ in F.

Problem 1 Let E and F be two Riesz spaces. Under what conditions can it be said $L_{so}(E,F) = L_n(E,F) \cap L_{uo}(E,F)$?

Proposition 2 *Let E, F, G be Riesz spaces. Then we have the following assertions.*

1. *If $T \in L_{so}(E,F)$ and $S \in L_n(F,G)$, then $ST \in L_{so}(E,G)$. As a consequence, $L_{so}(E)$ is a left ideal for $L_n(E)$. Similarly, $L_{\sigma-so}(E)$ is a left ideal for $L_c(E)$.*
2. *If $T \in L_{uo}(E,F)$ and $S \in L_{so}(F,G)$, then $ST \in L_{so}(E,G)$.*
3. *If $T \in L_{so}(E,E)$, then $T^n \in L_{so}(E,E)$ for all $n \in \mathbb{N}$.*
4. *If E is σ-Dedekind complete and σ-laterally complete and $S \in L_c(E,F)$ and $T \in L_{\sigma-so}(F,G)$, then $TS \in L_{\sigma-so}(E,G)$. In this case, $L_{\sigma-so}(E,F) = L_c(E,F)$.*

Proof 1. Let (x_α) be a net in E such that $x_\alpha \xrightarrow{uo} 0$. By assumption, $Tx_\alpha \xrightarrow{o} 0$. So, $STx_\alpha \xrightarrow{o} 0$. Hence, $ST \in L_{so}(E,G)$.

2. Let (x_α) be a net in E such that $x_\alpha \xrightarrow{uo} 0$. By assumption, $Tx_\alpha \xrightarrow{uo} 0$. So, $STx_\alpha \xrightarrow{o} 0$. Therefore, $ST \in L_{so}(E,G)$.
3. Let (x_α) be a net in E such that $x_\alpha \xrightarrow{uo} 0$. By assumption, $Tx_\alpha \xrightarrow{o} 0$ and so $Tx_\alpha \xrightarrow{uo} 0$. Therefore, $T^2x_\alpha \xrightarrow{o} 0$. Hence, $T^2 \in L_{so}(E,E)$. By induction, $T^n \in L_{so}(E,E)$ for all $n \in \mathbb{N}$.
4. Let E be a σ-Dedekind complete and σ-laterally complete Riesz space. By Theorem 3.9 of [7], we see that a sequence (x_n) in E is *uo*-null if and only if it is order null. So, if (x_n) be a sequence in E such that $x_n \xrightarrow{uo} 0$, then $x_n \xrightarrow{o} 0$. Thus, $Sx_n \xrightarrow{o} 0$ and then $TSx_n \xrightarrow{o} 0$. Hence, $TS \in L_{\sigma-so}(E,G)$. Clearly, we have $L_{\sigma-so}(E,F) = L_c(E,F)$. This ends the proof.

Let $T\colon E \to F$ be a positive operator between two Riesz spaces. We say that an operator $S\colon E \to F$ is dominated by T (or that T dominates S) whenever $|Sx| \le T|x|$ holds for each $x \in E$.

Theorem 1 *The following assertions are true.*

1. *If a positive so-continuous operator $T\colon E \to F$ dominates S, then S is so-continuous.*
2. *If E and F are Archimedean laterally complete Riesz spaces, G is order dense in Dedekind complete Riesz space E and $T\colon G \to F$ is order continuous lattice homomorphism, then T is σ-so-continuous.*

Proof 1. Let $T\colon E \to F$ be a positive *so*-continuous operator between two Riesz spaces such that T dominates $S\colon E \to F$ and let $x_\alpha \xrightarrow{uo} 0$ in E. It is obvious that $|x_\alpha| \xrightarrow{uo} 0$. So, by assumption, $T|x_\alpha| \xrightarrow{o} 0$ and from the inequality $|Sx| \le T|x|$, we have $Sx_\alpha \xrightarrow{o} 0$. Hence, S is *so*-continuous.

2. By Theorem 2.32 of [2], the formula

$$S(x) = \sup\{T(y) : y \in G \ and \ 0 \leq y \leq x\}, \quad x \in E^+,$$

defines an extension of T from E to F, which is an order continuous lattice homomorphism. Let $(x_n) \subseteq G$ is a *uo*-null sequence. By Theorem 3.10 of [7], (x_n) is order-null. Since S is order continuous, therefore $T(x_n) = S(x_n) \xrightarrow{o} 0$ in F.

Recall from [4] that $f \in E^*$ is said to be *un*-continuous, if for each *un*-null net $(x_\alpha) \subseteq E$, we have $f(x_\alpha) \to 0$ in $\mathbb{R}$.
It is clear that if E is an atomice Banach lattice with order continuous norm, then by Theorem 5.3 of [4], each $f \in E^*$ is σ-*so*-continuous iff it is a σ-*un*-continuous.

Remark 2 1. Let E be a atomic Banach lattice with order continuous norm. If $f : E \to \mathbb{R}$ is a positive σ-*so*-continuous, then $f = \lambda_1 f_{a_1} + \lambda_2 f_{a_2} + ... + \lambda_n f_{a_n}$, where $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}$ and $a_1, a_2, ..., a_n$ are atoms. Let $(x_n) \subseteq E$ be a *un*-null net. Because E is atomic with order continuous norm, by Theorem 5.3 of [4], (x_n) is *uo*-null. by assumption we have $f(x_n) \xrightarrow{o} 0$ and therefore it is norm-null. By Corollary 5.4 of [10], the proof is complete.
2. If E is non-atomice and $f : E \to \mathbb{R}$ is continuous and *so*-continuous, then by Corollary 5.4 of [10], $f = 0$.
3. By Corollary 2.6 of [12], $E^\sim_{uo}$ is an ideal of $E^\sim_n$ (or $E^\sim$) and so $E^\sim_{so}$ is an ideal of $E^\sim_n$.

Remark 3 Let E be a Banach lattice and such that $E^\sim_{uo}$ separates the points of E. By Proposition 2.13 of [12], the following conditions are equivalent.

1. E is finite dimension space.
2. $E^\sim_{so} = E^\sim_n$.
3. $E^\sim_{so}$ is an band of $E^\sim$.

Theorem 2 *For an order bounded linear functional f on a Riesz space E the following statements are equivalent.*

1. *f is so-continuous.*
2. *f^+ and f^- are both so-continuous.*
3. *$|f|$ is so-continuous.*

Proof (1) $\Rightarrow$ (2) Let $(x_\alpha) \subseteq E^+$ and $x_\alpha \xrightarrow{uo} 0$. Let (r_α) be a net in $\mathbb{R}$ such that $r_\alpha \downarrow 0$. According to Proposition 3.1 of [7], in view of $f^+x = \sup\{fy : 0 \leq y \leq x\}$, there exists a net (y_α) in E with $0 \leq y_\alpha \leq x_\alpha$ for each α and $f^+x_\alpha - r_\alpha \leq fy_\alpha$. So, $f^+x_\alpha \leq fy_\alpha + r_\alpha$. Since $x_\alpha \xrightarrow{uo} 0$, we have $y_\alpha \xrightarrow{uo} 0$. Thus, by assumption, $fy_\alpha \xrightarrow{o} 0$. It follows from $f^+x_\alpha \leq (fy_\alpha + r_\alpha) \xrightarrow{o} 0$ that $f^+x_\alpha \xrightarrow{o} 0$. Hence, f^+ is *so*-continuous. Now, as $f^- = (-f)^+$, we conclude that f^- is also *so*-continuous.
(2) $\Rightarrow$ (3) Follows from the identity $|f| = f^+ + f^-$.
(3) $\Rightarrow$ (1) Follows immediately from Theorem 1 by observing that $|f|$ dominates f.

Remark 4 One can easily formulate by himself the analogue of Theorem 2 for σ-*so*-continuous operators.

Recall that a subset A of a Riesz space is said to be order closed whenever $(x_\alpha) \subseteq A$ and $x_\alpha \xrightarrow{o} x$ imply $x \in A$. An order closed ideal is referred to as a band. Thus, an ideal A is a band if and only if $(x_\alpha) \subseteq A$ and $0 \leq x_\alpha \uparrow x$ imply $x \in A$. In the next theorem we show that $E^{\sim}_{so}$ and $E^{\sim}_{\sigma-so}$ are both bands of $E^{\sim}$. The details follow.

Theorem 3 *If E is a Riesz space, then $E^{\sim}_{so}$ and $E^{\sim}_{\sigma-so}$ are both bands of $E^{\sim}$.*

Proof We only show that $E^{\sim}_{so}$ is a band of $E^{\sim}$. That $E^{\sim}_{\sigma-so}$ is a band can be proven in a similar manner. Note first that if $|g| \leq |f|$ holds in $E^{\sim}$ with $f \in E^{\sim}_{so}$, then from Theorems 3.1 and 3.2 it follows that $g \in E^{\sim}_{so}$). That is $E^{\sim}_{so}$ is an ideal of $E^{\sim}$. To see that the ideal $E^{\sim}_{so}$ is a band, let $0 \leq f_\lambda \uparrow f$ in $E^{\sim}$ with $(f_\lambda) \subset E^{\sim}_{so}$, and let $0 \leq x_\alpha \xrightarrow{uo} 0$ in E. Then for each fixed λ we have

$$0 \leq f(x_\alpha) = ((f - f_\lambda)(x_\alpha) + f_\lambda(x_\alpha)) \xrightarrow{o} 0.$$

So, $f(x_\alpha) \xrightarrow{o} 0$. Thus, $f \in E^{\sim}_{so}$, and the proof is finished.

3 Order-to-unbounded Order Continuous Operators

Definition 2 An operator $T\colon E \to F$ between two Riesz spaces is said to be:

1. order-to-unbounded order continuous (for short, *ouo*-continuous), if $x_\alpha \xrightarrow{o} 0$ in E implies $Tx_\alpha \xrightarrow{uo} 0$ in F for each net $(x_\alpha) \subseteq E$.
2. σ-order-to-unbounded order continuous (for short, σ-*ouo*-continuous), if $x_n \xrightarrow{o} 0$ in E implies $Tx_n \xrightarrow{uo} 0$ in F for each sequence $(x_n) \subseteq E$.

The collection of all *ouo*-continuous (resp. σ-*ouo*-continuous) operators from E into F will be denoted by $L_{ouo}(E, F)$, (resp. $L_{\sigma-ouo}(E, F)$).

It is obvious that each identity operator on Riesz space E is an *ouo*-continuous operator and also we have $f \in E^{\sim}$ if and only if $f \in E^{\sim}_{ouo}$.

Theorem 4 *Let E be a normed Riesz space with order continuous norm and F be an atomic Banach lattice with order continuous norm, then each continuous operator $T\colon E \to F$ is σ-ouo-continuous.*

Proof Let $(x_n) \subseteq E$ be an order-null net. Since E has order continuous norm, (x_n) is a norm-null net. By continuity of T, we have, $(T(x_n))$ is norm-null and hence it is *un*-null. Because F is atomic with order continuous norm, by Theorem 5.3 of [4], $(T(x_n))$ is *uo*-null.

On the other hand, if $T\colon E \to F$ is *ouo*-continuous, it follows that T is a σ-*ouo*-continuous operator. However, the converse is not necessarily true. An example illustrating this point is given in Example 1.55 on page 46 of [2], where a σ-*ouo*-continuous operator is presented that is not *ouo*-continuous. It should

be noted that the operator T in Example 1.55 of [2] is σ-order continuous, and therefore σ-*ouo*-continuous. However, it is not *ouo*-continuous, as can be easily verified.
The following example shows that in general each *ouo*-continuous operator is not *uo*or *so*-continuous.

Example 4
The functional $f\colon \ell^1 \to \mathbb{R}$ defined by

$$f((x_1, x_2, ...)) = \sum_{i=1}^{\infty} x_i,$$

is *ouo*-continuous. Let $(x_\alpha) \subseteq E$ be an order-null net. Since ℓ^1 has order continuous norm, therefore (x_α) is norm-null and so $f(x_\alpha) \to 0$ in $\mathbb{R}$. On the other hands, $(e_n) \subseteq \ell^1$ is *uo*-null. But $(f(e_n))$ is not *uo*-null in $\mathbb{R}$. Hence f is not a *uo*-continuous operator.
The identity operator $I\colon \ell^1 \to \ell^1$ is *ouo*-continuous. Consider $(e_n) \subseteq \ell^1$ is *uo*-null, but it is not order-null in ℓ^1. Therefore $I\colon \ell^1 \to \ell^1$ is not *so*-continuous.

Theorem 5 *Every continuous operator from $C[0, 1]$ to ℓ^1 is σ-ouo-continuous.*

Proof Let $T\colon C[0, 1] \to \ell^1$ is a continuous operator. By Exercise 3 of page 313 of [2], T is a compact operator. Since $C[0, 1]^*$ has order continuous norm, by Theorem 5.44 of [2], there exist a reflexive Banach lattice F, the lattice homomorphism Q and compact operator S that $T = S \circ Q$. Let $(x_n) \subseteq C[0, 1]$ be an *o*-null sequence. Because Q is lattice homomorphism and therefore is order continuous, so $(Q(x_n))$ is *o*-null in F. F is a reflexive, so it has order continuous norm. Therefore $(Q(x_n))$ is norm-null in F. By continiuty of S, we have $(S(Q(x_n)))$ is norm-null and therefore is *un*-null in ℓ^1. Since ℓ^1 is atomic with order continuous norm, by Theorem 5.3 of [4], $T(x_n) = (S(Q(x_n)) \xrightarrow{uo} 0$ in ℓ^1.

In the following, we provide examples of new classifications of operators.

Example 5 1. Since, $L_1[0, 1]$ has order continuous norm and c_0 is an atomic Banach lattice with order continuous norm, the operator $T\colon L_1[0, 1] \to c_0$, given by

$$T(f) = (\int_0^1 f(x) sinx dx, \int_0^1 f(x) sin2x dx, \dots),$$

is a σ-*ouo*-continuous operator.

2. The operator $T\colon C[0, 1] \to \ell^1$, given by

$$T(f) = (\frac{\int_0^1 f(x) \sin x dx}{n^2}, \frac{\int_0^1 f(x) \sin 2x dx}{n^2}, \dots)$$

is a σ-*ouo*-continuous operator.

3. Let B be a projection band of Riesz space E and P_B the corresponding band projection. It follows easily from $0 \leq P_B \leq I$ (see Theorem 1.44 of [2]) that if $x_\alpha \xrightarrow{o} 0$ in E then $P_B x_\alpha \xrightarrow{o} 0$ in B and therefore $P_B x_\alpha \xrightarrow{uo} 0$ in B. So P_B is an *ouo*-continuous operator.
4. Let $E^\sim$ be the order dual of Riesz space E. It is obvious that each $f \in E^\sim$ is a *ouo*-continuous operator.

Remark 5 1. Let E, F be two Riesz spaces such that E is finite-dimensional. Then $L_{uo}(E,F) = L_{ouo}(E,F)$ and $L_{\sigma-uo}(E,F) = L_{\sigma-ouo}(E,F)$.
2. If $T\colon E \to F$ is an *so*-continuous operator and $S\colon F \to G$ is *ouo*-continuous, it is obvious that $S \circ T\colon E \to G$ is an *uo*-continuous operator.
3. If $T\colon E \to F$ is an *ouo*-continuous operator and $S\colon F \to G$ is *so*-continuous, it is obvious that $S \circ T\colon E \to G$ is an *o*-continuous operator.
4. If $T\colon E \to F$ is an *ouo*-continuous operator and $S\colon F \to G$ is *uo*-continuous, it is obvious that $S \circ T\colon E \to G$ is an *ouo*-continuous operator.
5. Let G be a sublattice of Dedekind complete Riesz space E. Then $T\colon E \to F$ is *ouo*-continuous if and only if $T|_G$ is *ouo*-continuous.
6. Let $T, S\colon E \to F$ be two operators and $0 \leq T \leq S$. If S is *ouo*-continuous, then T is an *ouo*-continuous operator.

Since, the proofs of the three following theorems are straightforward, we will not provide them here.

Theorem 6 *Let E and F be two Riesz spaces that F is order continuous and atomic. An operator $T\colon E \to F$ is σ-ouo-continuous if and only if σ-oun-continuous operator.*

Theorem 7 *Let E and F be two Banach lattices. Then, by one of the following assertions, $T\colon E \to F$ is an ouo-continuous operator.*

1. *T is order continuous,*
2. *T is uo-continuous,*
3. *T is so-continuous.*

Theorem 8 *1. Let $T\colon E \to F$ be an order bounded operator between two Riesz spaces with F Dedekind complete. If T is an uo-continuous operator, then T, T^+, T^- and $|T|$ are ouo-continuous operators.*
2. If $T \in L_{uo}(E,E)$, then $T^n \in L_{ouo}(E,E)$ for all $n \in \mathbb{N}$.

Theorem 9 *Let E and F be two Riesz spaces that F is a Dedekind complete. An operator $0 \leq T\colon E \to F$ is ouo-continuous if and only if $x_\alpha \downarrow 0$ in E implies $T(x_\alpha) \downarrow 0$.*

Proof Let T be an *ouo*-continuous operator and $(x_\alpha) \subseteq E$ with $x_\alpha \downarrow 0$ in E. Because $x_\alpha \xrightarrow{o} 0$ by assumption we have $T(x_\alpha) \xrightarrow{uo} 0$. On the other hand $T(x_\alpha) \downarrow z$ and therefore $T(x_\alpha) \xrightarrow{uo} z$. Since *uo*-convergence are unique, we have $z = 0$.

Conversely, now let $(x_\alpha) \subseteq E$ be an *o*-null net. there exists another net (y_β) in E such that $y_\beta \downarrow 0$ and that for every β, there exists α_0 such that $|x_\alpha| \leq y_\beta$

for all $\alpha \geq \alpha_0$. By assumption, we have $T(y_\beta) \downarrow 0$. So $|T(x_\alpha)| \leq T|x_\alpha| \leq T(y_\beta)$. It means that $T(x_\alpha) \xrightarrow{o} 0$ and hence $T(x_\alpha) \xrightarrow{uo} 0$ in F.

Corollary 1 *If F is Dedekind complete Riesz space and $T: E \to F$ is a positive operator, then T is order continuous if and only if it is ouo-continuous.*

Corollary 2 *Let E and F be two Archimedean Riesz spaces that F is a Dedekind complete. An operator $0 \leq T: E \to F$ is ouo-continuous if and only if there is an order dense and topologically majorizing sublattice H such that $T|_H$ is ouo-continuous.*

Proposition 3 *If $T: E \to F$ is a so-continuous operator, then its order adjoint $T': F^\sim \to E^\sim$ is ouo-continuous.*

Proof Let $T: E \to F$ be a *so*-continuous operator. It is obvious that it is an order continuous operator. By Lemma 1.54 of [2], T is an order bounded operator. Now by Theorem 1.73 of [2], its order adjoint $T': F^\sim \to E^\sim$ is order continuous. Therefore by Remark 7, T' is an *ouo*-continuous operator.

Remark 6 The converse of Proposition 3, is not true in general. Consider the identity operator $I: c_0 \to c_0$. Its order adjoint $I: \ell^1 \to \ell^1$ is *ouo*-continuous, while $I: c_0 \to c_0$ is not *so*-continuous.

Theorem 10 *Let $T: E \to F$ be an operator between to Riesz spaces. Then there exist a vector lattice G, an operator $T_1: E \to G$ and an operator $T_2: G \to F$ that $T = T_2 \circ T_1$. Such that*

1. *T_1 is ouo-continuous.*
2. *T is so-continuous if T_2 is so-continuous.*
3. *T is ouo-continuous if T_2 is ouo-continuous.*

Proof Let $T: E \to F$ be an operator and $(x_\alpha) \subseteq E$ be a *uo*-null net. We have for all $u \in E^+$, $(|x_\alpha| \wedge u)$ is *o*-null. Let $u \in E^+$ is an arbitrary vector and B_u be a band generated by u in E. We put $G = B_u$ and $T_1: E \to G$ by $T_1(x) = P_G(x)$, where P_G is band projection from E to G. It is clear that T_1 is well define and it is an *ouo*-continuous operator.

We put $T_2: G \to F$ by $T_2(z) = T_2(P_G x) = T(x)$ that $z \in G$. T_2 is well define and we have $T = T_2 \circ T_1$. Let $(x_\alpha) \subseteq E$ be a *uo*-null. Therefore $(P_G(x_\alpha))$ is *uo*-null. Now if T_2 is *so*-continuous, we have $T(x_\alpha) = T_2(P_G(x_\alpha)) \xrightarrow{o} 0$. So T is *so*-continuous. The same way, if T_2 is *ouo*-continuous, then T is an *ouo*-continuous operator.

Proposition 4 *Let $T: E \to E$ be an operator. The following assertions are equivalent.*

1. E has finite dimensional.
2. T is *so*-continuous if and only if is *ouo*-continuous.

Proof 1 $\Rightarrow$ 2 Let E has finite dimentional, it is clear that $T\colon E \to E$ is a *so*-continuous operator if and only if it is an *ouo*-continuous operator.

2 $\Rightarrow$ 1 Conversely, let $T\colon E \to E$ is *so*-continuous if and only if it is an *ouo*-continuous operator. Suppose E has infinite dimensional. Therefore there exists a net $(x_\alpha) \subseteq E$ that it is *uo*-null while it is not *o*-null. It is a contradiction by assumption.

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