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A Numrical Method for Solving a Parabolic Problem Emanating in Financial Mathematics

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Abstract This study aims to develop a robust numerical algorithm for solving parabolic partial differential equations (PDEs) arising in the domain of financial mathematics. The proposed approach leverages the finite difference method (FDM) to discretize the temporal and spatial domains of the problem. To approximate the unknown solution, we employ a polynomial interpolation technique, ensuring high accuracy and stability in the numerical solution. The effectiveness and efficiency of our method are demonstrated through comprehensive numerical experiments, showcasing its potential for practical applications in financial modeling.

Keywords Parabolic problem · Financial mathematics · Polynomial function · Finite difference method

Mathematics Subject Classification (2010) 91G80 · 35K55

1 Introduction

In recent years, there has been increasing interest in problems arising in financial mathematics, particularly in option pricing. The standard approach to this problem involves studying equations of a parabolic type.

Parabolic partial differential equations (PDEs), such as the Black-Scholes equation, are extensively used in financial mathematics for pricing options and modeling various derivative securities. These equations help in understanding the dynamics of financial markets and making informed investment decisions. Here are some key applications of parabolic PDEs in financial mathematics:

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Option Pricing: The Black-Scholes equation, a parabolic PDE, is foundational for pricing European-style options. It considers factors such as the underlying asset price, time to expiration, strike price, risk-free interest rate, and volatility to determine the fair value of an option.
Risk Management: Parabolic PDEs are also utilized in risk management models to evaluate the risk exposure of financial portfolios. By simulating various scenarios and analyzing potential outcomes, financial institutions can better hedge against adverse market movements.
Interest Rate Models: Parabolic PDEs are employed in modeling interest rate derivatives and determining the term structure of interest rates. Models such as the Heath-Jarrow-Morton framework use parabolic PDEs to describe the evolution of interest rates over time and across different maturities.
Credit Risk Modeling: Parabolic PDEs are applied in credit risk models to assess the probability of default and the associated losses in credit portfolios. These models help financial institutions gauge their credit risk exposure and allocate capital accordingly.
Portfolio Optimization: Parabolic PDEs can be used in portfolio optimization models to find the optimal allocation of assets that maximizes expected returns while minimizing risk. By solving these equations, investors can construct well-balanced portfolios tailored to their risk preferences.
Overall, parabolic PDEs play a crucial role in financial mathematics by providing mathematical frameworks to analyze and understand various financial phenomena, ultimately aiding in decision-making processes within the financial industry.
Parabolic problems appear in many important scientific and technological fields. Hence, the analysis, design, implementation, and testing of inverse algorithms are of great scientific and technological interest. To date, various methods have been developed for analyzing parabolic problems based on measured temperatures inside the material ([1]–[10]). Beck and Murio [3], presented a new method that combines the function specification method of Beck with the regularization technique of Tikhonov. Beck et al. [2] compare the FSM, the Tikhonov regularization and the iterative regularization, using experimental data. Pourgholi et al. [5], presented a stable solution for a parabolic problem and proved the existence, uniqueness, and stability of the solution. Another technique to solve parabolic problems is developed based on the use of the solution to the auxiliary problem as a basis function [6]. Many numerical and experimental methods have been developed for solving parabolic problems ([4,7–13]).
In this paper, the attention has been focused on the numerical solution of parabolic problems emanating from financial mathematics.

2 Formulating the problem

In this section, we consider the following parabolic problem,

$$\frac{\partial T(x,t)}{\partial t} - g(x,t) = (\frac{\partial}{\partial x}k(x,t))\frac{\partial T(x,t)}{\partial x} + k(x,t)\frac{\partial^2 T(x,t)}{\partial x^2}, \quad 0 < x < 1,\ t > 0, \tag{1a}$$

$$T(x,0) = \varphi(x), \quad 0 \le x \le 1, \tag{1b}$$

$$k(0,t)\frac{\partial T(0,t)}{\partial x} = q(0,t), \quad t > 0, \tag{1c}$$

$$k(1,t)\frac{\partial T(1,t)}{\partial x} = q(1,t), \quad t > 0, \tag{1d}$$

and the overspecified condition

$$T(x_i,t) = p_i(t), \quad 0 \le x_i \le 1,\ (i = 0,\dots,s),\ t > 0, \tag{1e}$$

where $\varphi(x)$, the heat flux $q(0,t)$, and the heat flux $q(1,t)$ are continuous known functions, while the temperature $T(x,t)$, and the thermal conductivity $k(x,t)$ are unknown which remain to be determined from some interior temperature measurements (the known overspecified conditions).
In the next section we present a semi-discrete numerical algorithm to solve this problem.

3 Numerical scheme for the parabolic problem

To solve the parabolic problem (1), we consider a polynomial function to approximate temperature distribution. Since, $s+1$ over-specified conditions (1e) are available, we chose a (s)-degree polynomial as below:

$$T(x,t) = a_0(t) + a_1(t)x + \cdots + a_s(t)x^s. \tag{2}$$

$(s+1)$ time-dependent coefficients of polynomial are determined from conditions (1e) as follows:
$a_0(t), a_1(t), \dots, a_s(t)$ can be obtained from the following linear matrix equation

$$\Lambda x = b,$$

where

$$\Lambda = \begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^s \\ 1 & x_1 & x_1^2 & \dots & x_1^s \\ . & . & . & & . \\ . & . & . & & . \\ . & . & . & & . \\ 1 & x_s & x_s^2 & \dots & x_s^s \end{pmatrix},\ b = \begin{pmatrix} p_0(t) \\ p_1(t) \\ . \\ . \\ . \\ p_s(t) \end{pmatrix},\ x = \begin{pmatrix} a_0(t) \\ a_1(t) \\ . \\ . \\ . \\ a_s(t) \end{pmatrix}. \tag{3}$$

Furthermore, the parabolic conduction equation (1a) can be written as

$$\frac{\partial k(x,t)}{\partial x}\frac{\partial T(x,t)}{\partial x}+k(x,t)\frac{\partial^2 T(x,t)}{\partial x^2}=\frac{\partial T(x,t)}{\partial t}-g(x,t). \tag{4}$$

For discretizing this equation, $k(x,t)$ approximate by the central finite difference. Therefore, the discretization of the equation (4) can be expressed as

$$\frac{k_{i+1,j}-k_{i-1,j}}{2\Delta x}A_{i,j}+k_{i,j}B_{i,j}=C_{i,j}, \tag{5}$$

where

$$A_{i,j}=(\frac{\partial T(x,t)}{\partial x})_{i,j}=a_1(t_j)+2a_2(t_j)x_i+\cdots+(s)a_s(t_j)x_i^{s-1}, \tag{6}$$

$$B_{i,j}=(\frac{\partial^2 T(x,t)}{\partial x^2})_{i,j}=2a_2(t_j)+\cdots+(s)(s-1)a_s(t_j)x_i^{s-2}, \tag{7}$$

$$\begin{aligned}C_{i,j}&=(\frac{\partial T(x,t)}{\partial t})_{i,j}-g_{i,j}\\&=\frac{da_0(t_j)}{dt}+\frac{da_1(t_j)}{dt}x_i+\frac{da_2(t_j)}{dt}x_i^2+\cdots+\frac{da_s(t_j)}{dt}x_i^s-g_{i,j},\end{aligned} \tag{8}$$

and $x_i=i\Delta x$, $(i=0,1,\ldots,N)$ and $t_j=j\Delta t$, $(j=0,1,\ldots,M)$. Equation (5) can be rearranged as

$$-A_{i,j}k_{i-1,j}+2\Delta xB_{i,j}k_{i,j}+A_{i,j}k_{i+1,j}=2\Delta xC_{i,j},\ i=1,2,\ldots,N-1. \tag{9}$$

$k(x,t)$ at boundary ends can be obtained as

$$k_{0,j}=\frac{q_{0,j}}{(\frac{\partial T(x,t)}{\partial x})_{0,j}}=\frac{q_{0,j}}{a_1(t_j)},\quad i=0, \tag{10}$$

and

$$k_{N,j}=\frac{q_{N,j}}{(\frac{\partial T(x,t)}{\partial x})_{N,j}}=\frac{q_{N,j}}{a_1(t_j)+2a_2(t_j)+\cdots+(s)a_s(t_j)},\quad i=N. \tag{11}$$

By using equations (9)-(11), we obtain the following linear algebraic system of equations

$$AK=B, \tag{12}$$

where

$$A=\begin{pmatrix}2\Delta xB_{1,j} & A_{1,j} & 0 & 0 & 0 & 0 & 0\\ -A_{2,j} & 2\Delta xB_{2,j} & A_{2,j} & 0 & 0 & 0 & 0\\ . & . & . & . & . & . & .\\ . & . & . & . & . & . & .\\ . & . & . & . & . & . & .\\ 0 & 0 & 0 & 0 & -A_{N-2,j} & 2\Delta xB_{N-2,j} & A_{N-2,j}\\ 0 & 0 & 0 & 0 & 0 & -A_{N-1,j} & 2\Delta xB_{N-1,j}\end{pmatrix},$$

$$B = \begin{pmatrix} 2\Delta x C_{1,j} + A_{1,j} k_{0,j} \\ 2\Delta x C_{2,j} \\ . \\ . \\ . \\ 2\Delta x C_{N-2,j} \\ 2\Delta x C_{N-1,j} - A_{N-1,j} k_{N,j} \end{pmatrix}, \; K = \begin{pmatrix} k_{1,j} \\ k_{2,j} \\ . \\ . \\ . \\ k_{N-2,j} \\ k_{N-1,j} \end{pmatrix}.$$

4 Stability

Theorem 1 *Linear algebraic system* (12)*, is stable for*

$$\Delta x \le \frac{|a_1(t_j)| + 2|a_2(t_j)| + \cdots + (s)|a_s(t_j)| - 0.5}{2 \times |a_2(t_j)| + \cdots + s(s-1)|a_s(t_j)|}.$$

Proof From equation (12) we obtain

$$K = A^{-1}B. \tag{13}$$

In system (13), the matrix determining the propagation of the error is A^{-1}. Therefore, scheme (12) will be stable when the modulus of every eigenvalue of A^{-1} does not exceed one. Application of Gerschgorins circle theorem to the matrix A shows that its eigenvalues λ lie on or within the circle

$$|\lambda - 2\Delta x B_{1,j}| \le |A_{1,j}|, \tag{14}$$

by substituting (6) and (7) into (14) we obtain

$$|\lambda| \ge 2\Delta x|B_{1,j}| - |a_1(t_j)| - 2|a_2(t_j)| - \cdots - (s)|a_s(t_j)|. \tag{15}$$

Similarly we require

$$|\lambda| \ge 2\Delta x|B_{N-1,j}| - |a_1(t_j)| - 2|a_2(t_j)| - \cdots - (s)|a_s(t_j)|. \tag{16}$$

For rows $2, 3, \ldots, N-2$,

$$|\lambda| \ge 2\Delta x|B_{i,j}| - 2|a_1(t_j)| - 2 \times 2|a_2(t_j)| - \cdots - 2 \times (s)|a_s(t_j)|. \tag{17}$$

For overall stability, from (15)-(17), we obtain

$$\Delta x \le \frac{|a_1(t_j)| + 2|a_2(t_j)| + \cdots + (s)|a_s(t_j)| - 0.5}{2 \times |a_2(t_j)| + \cdots + s(s-1)|a_s(t_j)|}.$$

Table 1 The values of $T(x,t)$ when $\Delta x = 0.01$, for Example 1.

	Exact	*Numerical*	*Exact*	*Numerical*	*Exact*	*Numerical*
x	$T(x,0.4)$	$T(x,0.4)$	$T(x,0.45)$	$T(x,0.45)$	$T(x,0.5)$	$T(x,0.5)$
0.0	0.0965	0.0965	0.1033	0.1033	0.1092	0.1092
0.1	0.0670	0.0670	0.0717	0.0717	0.0758	0.0758
0.2	0.0429	0.0429	0.0459	0.0459	0.0485	0.0485
0.3	0.0241	0.0241	0.0258	0.0258	0.0273	0.0273
0.4	0.0107	0.0107	0.0115	0.0115	0.0121	0.0121
0.5	0.0027	0.0027	0.0029	0.0029	0.0030	0.0030
0.6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.7	0.0027	0.0027	0.0029	0.0029	0.0030	0.0030
0.8	0.0107	0.0107	0.0115	0.0115	0.0121	0.0121
0.9	0.0241	0.0241	0.0258	0.0258	0.0273	0.0273
1.0	0.0429	0.0429	0.0459	0.0459	0.0485	0.0485

Table 2 The values of $k(x,t)$ when $\Delta x = 0.01$, for Example 1.

	Exact	*Numerical*	*Exact*	*Numerical*	*Exact*	*Numerical*
x	$k(x,0.4)$	$k(x,0.4)$	$k(x,0.45)$	$k(x,0.45)$	$k(x,0.5)$	$k(x,0.5)$
0.0	0.6189	0.6189	0.5289	0.5289	0.4520	0.4520
0.1	0.4332	0.4485	0.3702	0.3836	0.3164	0.3280
0.2	0.1227	0.1341	0.1049	0.1148	0.0896	0.0983
0.3	0.1255	0.1336	0.1073	0.1144	0.0917	0.0979
0.4	0.4409	0.4466	0.3768	0.3818	0.3221	0.3264
0.5	0.6299	0.6330	0.5383	0.5411	0.4600	0.4625
0.6	0.4222	0.4222	0.3608	0.3608	0.3084	0.3084
0.7	0.0919	0.0992	0.0785	0.0848	0.0671	0.0725
0.8	0.0805	0.0957	0.0688	0.0819	0.0588	0.0701
0.9	0.3894	0.4130	0.3328	0.3532	0.2844	0.3020
1.0	0.5792	0.5792	0.4950	0.4950	0.4231	0.4231

5 Numerical Results and Discussion

In this section, we are going to demonstrate some numerical results for $T(x,t)$, and $k(x,t)$ in the parabolic problem (1)-(5). All the computations are performed on the PC (pentium(R) 4 CPU 3.20 GHz).

Example 1 In this example we will solve the problem (1) with the given data [4],

$$T(x,0) = 0, \qquad 0 \le x \le 1, \tag{18}$$

$$q(0,t) = -1.2(2 + 0.25e^{-0.36})te^{-t(\pi+1)}, \qquad t > 0, \tag{19}$$

$$q(1,t) = 0.8(2 + 0.25e^{-1.96})te^{-t(\pi+1)}, \qquad t > 0, \tag{20}$$

$$T(0,t) = 0.36te^{-t}, \qquad t > 0, \tag{21}$$

$$T(0.5,t) = 0.01te^{-t}, \qquad t > 0, \tag{22}$$

$$T(1,t) = 0.16te^{-t}, \qquad t > 0, \tag{23}$$

$$g(x,t) =(x - 0.6)^2(1 - t)e^{-t} - 2te^{-t(\pi+1)}\{[1 + 0.25e^{-4(x-0.3)^2} + \cos(4\pi x)]$$

$$+ (x-0.6)[0.25(-8x+2.4)e^{-4(x-0.3)^2} - 4\pi\sin(4\pi x)]\}, \quad 0 \le x \le 1. \tag{24}$$

The exact solutions of this problem is

$$T(x,t) = (x-0.6)^2 te^{-t},$$
$$k(x,t) = e^{-\pi t}[1 + 0.25e^{-4(x-0.3)^2} + \cos(4\pi x)].$$

To solve this example, since three over-specified condition (21)-(23) are available, we chose a 2-degree polynomial as below:

$$T(x,t) = a_0(t) + a_1(t)x + a_2(t)x^2.$$

Furthermore, since $a_1(t) = -\frac{6}{5}te^{-t}$ and $a_2(t) = te^{-t}$, then $\Delta x = 0.01$.
Tables 1 and 2, show the values of $T(x,t)$ and $k(x,t)$ for the problem (1) with the given data (18)-(24), and Figs. 1 and 2, show the comparison between the exact results and the present numerical results for $T(x,t)$ and $k(x,t)$ when $\Delta x = 0.01$.

Example 2 In this example we will solve the problem (1) with the given data [4],

$$T(x,0) = \sin(x), \qquad 0 \le x \le 1, \tag{25}$$
$$q(0,t) = 0.3e^{-2t}, \qquad t > 0, \tag{26}$$
$$q(1,t) = \frac{2}{3}e^{-2t}\cos(1), \qquad t > 0, \tag{27}$$
$$T(0,t) = 0, \qquad t > 0, \tag{28}$$
$$T(0.5,t) = e^{-t}\sin(0.5), \qquad t > 0, \tag{29}$$
$$T(1,t) = e^{-t}\sin(1), \qquad t > 0, \tag{30}$$

$$g(x,t) = \begin{cases} -e^{-t}\sin(x) + e^{-2t}[(0.3 + \frac{7x}{6})\sin(x) - \frac{7}{6}\cos(x)], & 0 \le x < 0.6, \\ -e^{-t}\sin(x) + e^{-2t}[(1.5 + \frac{5x}{6})\sin(x) + \frac{5}{6}\cos(x)], & 0.6 \le x \le 1. \end{cases} \tag{31}$$

The exact solutions of this problem is

$$T(x,t) = e^{-t}\sin(x),$$
$$k(x,t) = \begin{cases} (0.3 + \frac{7x}{6})e^{-t}, & 0 \le x \le 0.6, \\ (1.5 + \frac{5x}{6})e^{-t}, & 0.6 \le x \le 1. \end{cases}$$

To solve this example, since three over-specified condition (28)-(30) are available, we chose a 2-degree polynomial as below:

$$T(x,t) = a_0(t) + a_1(t)x + a_2(t)x^2.$$

Furthermore, since $a_1(t) = 1.0762e^{-t}$ and $a_2(t) = -0.2348e^{-t}$, then $\Delta x = 0.01$.
Figs. 3 and 4, are shown the values $T(x,t)$ and $k(x,t)$ for the problem (1) with the given data (25)-(31) when $\Delta x = 0.01$.

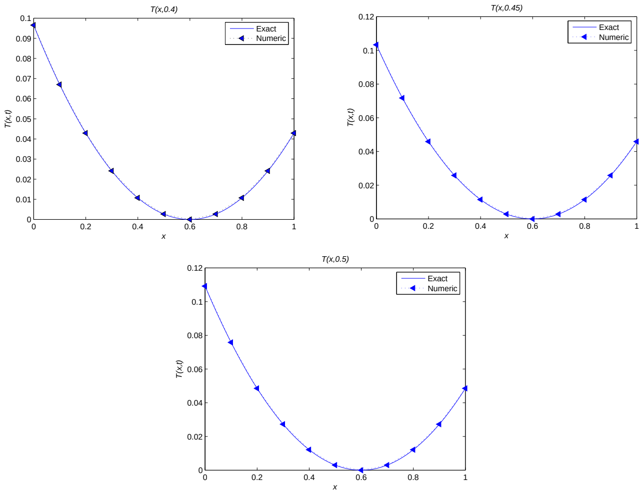

Fig. 1 Comparison between the exact and numerical results for $T(x, 0.4)$, $T(x, 0.45)$, and $T(x, 0.5)$, in Example 1, with the given data (18)-(24) when $\varDelta x = 0.01$.

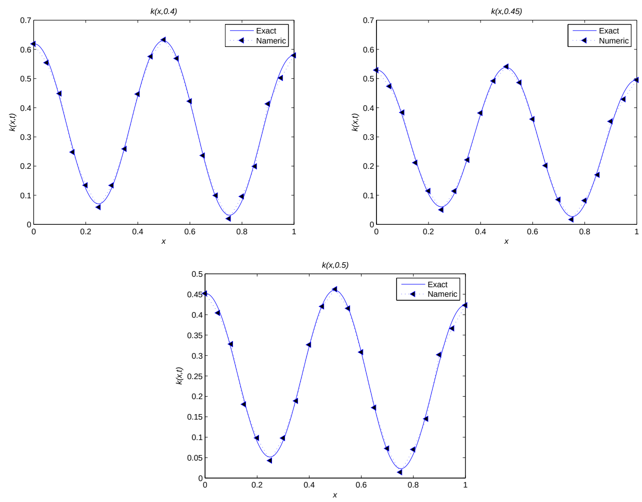

Fig. 2 Comparison between the exact and numerical results for $k(x, 0.4)$, $k(x, 0.45)$, and $k(x, 0.5)$, in Example 1, with the given data (18)-(24) when $\varDelta x = 0.01$.

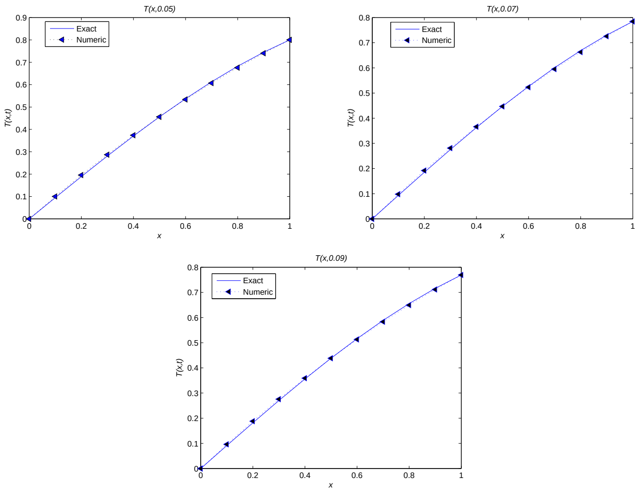

Fig. 3 Comparison between the exact and numerical results for $T(x, 0.05)$, $T(x, 0.07)$, and $T(x, 0.09)$, in Example 2, with the given data (25)-(31) when $\Delta x = 0.01$.

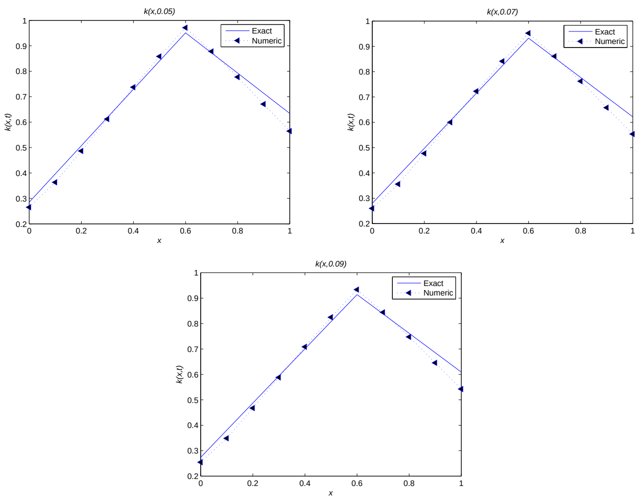

Fig. 4 Comparison between the exact and numerical results for $k(x, 0.05)$, $k(x, 0.07)$, and $k(x, 0.09)$, in Example 2, with the given data (25)-(31) when $\Delta x = 0.01$.

6 Conclusion

In this study, we have successfully applied a numerical method to address a parabolic partial differential equation within the context of financial mathematics. The results obtained from the illustrative example demonstrate that the proposed finite difference method, combined with polynomial interpolation, is both efficient and accurate in estimating the unknown function $k(x,t)$. Our numerical experiments confirm the reliability and precision of the method, indicating its potential utility for solving similar parabolic problems in various applications.

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