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Numerical Solutions of the Mechanical Vibrations via the Haar Wavelet Segmentation Method

Abdolreza Momeni · Kazem Nouri

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Abstract Due to the importance of fluctuating nonlinear differential equations in various branches of engineering, basic and applied sciences, various analytical and numerical methods have been used by researchers to solve such equations. Therefore, in this research, we have analyzed and investigated such equations and presented a useful method to find the approximate solutions of these equations, and we have compared the numerical results obtained from this method with their analytical or Runge-Kutta solutions.

Keywords Haar Wavelet · Differential Equations · Duffing · Non-Linear Oscillator · Segmentation Method

Mathematics Subject Classification (2010) 65T60 · 33E30 · 34C15

1 Introduction

The theory of wavelets was introduced to the field of science in the early 20th century by a scientist named Alfred Haar, and later this theory was developed and expanded by others. The simplest wavelet is the Haar wavelet, which is actually a first-order Daubechies wavelet.
Due to the simplicity and low cost and high speed of calculation as well as the sparse of the Haar matrix, using this wavelet in solving different types of differential equations is very easy and convenient and leads to acceptable results [1–8].

A. Momeni and K. Nouri
Department of Mathematics, Faculty of Mathematics, Statistics and Computer Sciences, Semnan University, P.O. Box 35195-363, Semnan, Iran.
Tel.: +98-23-31535768
Fax: +98-23-31535700
E-mail: a_momeni@semnan.ac.ir E-mail: knouri@semnan.ac.ir (Corresponding Author)

Here, we evaluate the approximate solution of a form of oscillating nonlinear equations as follows by using the Haar Wavelet Segmentation Method (HWSM) [9–11]:

$$f''(t) + p_0\left(f^2 - 1\right)f' + q_0 f + r_0 f^3 = s_0 r_e(t)\,;\; f(0) = f_0\;,\; f'(0) = f'_0,$$

where p_0, q_0, r_0 and s_0 are four real coefficients.

1.1 **Non-Linear Oscillator Equations**

We focus on numerical solution of the nonlinear relativistic harmonic oscillator equations in the following forms:

$$p_0 \frac{d^2 f}{dt^2} + \frac{q_0 f^m}{(1+f^2)^n} = s_0 r_e(t)\,;\quad m, n \in \mathbb{Q}, \tag{1}$$

with initial conditions $f(0) = f_0\;,\; f'(0) = f'_0$.

1.2 **Pendulum Equation**

We Analyze the pendulum equation

$$\theta'' + p_0 \sin\theta = 0, \tag{2}$$

where $p_0 = \frac{g}{l};\;\; g = 9.81\frac{m}{s^2}$ and l being the length of the pendulum (see Fig. 1).

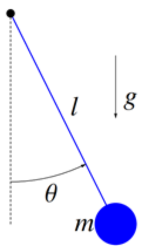

Fig. 1 Simple Pendulum

1.3 **Analytical Solution**

In first,by putting $(m = 3n = 3)$, in Equation (1), we have:

$$\frac{d^2 f}{dt^2} + \frac{q_0}{p_0}\frac{f^3}{f^2+1} = \frac{d^2 f}{dt^2} + \frac{q_0}{p_0}\left(\frac{f^3+f}{f^2+1} - \frac{f}{f^2+1}\right) = \frac{d^2 f}{dt^2} + \frac{q_0}{p_0}\left(f - \frac{f}{f^2+1}\right) = \frac{d^2 f}{dt^2} + g(f) = 0.$$

Now, according to the variational principle, we have

$$J\left(f\right)=\int_0^t\left(-\frac{1}{2}f'^2+G(f)\right)dt,$$

where $T=\frac{2\pi}{\omega}$ is period of Equation (1),

$$G\left(f\right)=\int g(f)df=\frac{q_0}{p_0}\left(\frac{1}{2}f^2-\frac{1}{2}\ln\left(f^2+1\right)\right).$$

So, its Hamiltonian is as follows (see [10]):

$$H=\frac{1}{2}f'^2+G\left(f\right)=G\left(f_0\right),$$

or

$$R\left(t\right)=\frac{1}{2}f'^2+G\left(f\right)-G\left(f_0\right)=0. \tag{3}$$

If we put the following trial function to get the angular frequency θ,

$$f\left(t\right)=f_0\cos\left(\theta t\right), \tag{4}$$

then, by inserting the above solution in Equation (3), we have

$$R\left(t\right)=\frac{1}{2}\theta^2 f_0^2\sin^2(\theta t)+G\left(f_0\cos\left(\theta t\right)\right)-G\left(f_0\right)=0.$$

Also, if we put $\theta t=\frac{\pi}{4}$, then we have

$$\begin{aligned}\theta&=\sqrt{\frac{2\left(G\left(f_0\right)-G\left(f\right)\right)}{f_0^2\sin^2(\theta t)}}=\pm\sqrt{\frac{2\left(G\left(f_0\right)-G\left(f_0\cos\left(\theta t\right)\right)\right)}{f_0^2\sin^2(\theta t)}}=\pm\sqrt{\frac{2\left(G\left(f_0\right)-G\left(\frac{\sqrt{2}}{2}f_0\right)\right)}{\frac{1}{2}f_0^2}}\\&=\pm\frac{2}{f_0}\sqrt{G\left(f_0\right)-G\left(\frac{\sqrt{2}}{2}f_0\right)}=\pm\frac{2}{f_0}\sqrt{\frac{q_0}{p_0}\left(\frac{1}{2}f_0^2-\frac{1}{2}ln(f_0^2+1)\right)-\frac{q_0}{p_0}\left(\frac{1}{4}f_0^2-\frac{1}{2}ln\left(\frac{1}{4}f_0^2+1\right)\right)}\\&=\pm\frac{2}{f_0}\sqrt{\frac{q_0}{p_0}\left(\frac{1}{4}f_0^2-\frac{1}{2}ln(f_0^2+1)\right)+\frac{1}{2}ln\left(\frac{1}{4}f_0^2+1\right)}=\pm\frac{2}{f_0}\sqrt{\frac{q_0}{p_0}\left(\frac{1}{4}f_0^2+\frac{1}{2}ln\frac{\frac{1}{4}f_0^2+1}{f_0^2+1}\right)}.\end{aligned}$$

Then its periodic function can be written as follows:

$$T=\frac{2\pi}{\theta}=\frac{2\pi}{\sqrt{\frac{2(G(f_0)-G(f))}{f_0^2\sin^2(\theta t)}}}\pm\frac{2\pi}{\frac{2}{f_0}\sqrt{G\left(f_0\right)-G\left(\frac{\sqrt{2}}{2}f_0\right)}}=\pm\frac{\pi f_0}{\sqrt{G\left(f_0\right)-G\left(\frac{\sqrt{2}}{2}f_0\right)}}.$$

Finally, by replacing θ into relation (4), we get

$$\begin{aligned}f\left(t\right)&=f_0\cos\left(\sqrt{\frac{2\left(G\left(f_0\right)-G\left(f\right)\right)}{f_0^2\sin^2(\theta t)}}t\right)=f_0\cos\left(\frac{2}{f_0}\sqrt{G\left(f_0\right)-G\left(\frac{\sqrt{2}}{2}f_0\right)}t\right)\\&=f_0\cos\left(\frac{2}{f_0}\sqrt{\frac{q_0}{p_0}\left(\frac{1}{4}f_0^2+\frac{1}{2}ln\frac{\frac{1}{4}f_0^2+1}{f_0^2+1}\right)}t\right).\end{aligned}$$

In the same way for Equation (2), we have

$$G\,(f)=\int g(f)df=p_0\int sinf df=-p_0\cos f,$$

and then

$$f\,(t)=f_0\cos\frac{2}{f_0}\sqrt{p_0\left(\cos f_0-\cos\frac{\sqrt{2}}{2}f_0\right)}t.$$

2 Haar Wavelets Family

The basis of the uniform Haar Wavelet family for each $x\in[A,B]=[0,1]$, and each $i\geq 1$, is defined as follows:

$$\mathfrak{h}_i(x)=\begin{cases}+1, & x\in[z_1,z_2),\\ -1, & x\in[z_2,z_3),\\ 0, & \text{otherwise,}\end{cases}$$

where

$$z_1=\frac{k}{\mu}\,,\,z_2=\frac{k+0.5}{\mu}\,,\,z_3=\frac{k+1}{\mu}.$$

In these formulate integer $\mu=2^j,\ j=0,1,\ldots,\mathfrak{J}$ indicates the level of the wavelet; $\kappa=0,1,\ldots,\mu-1$ is the translation parameter. The maximal level of resolution is $\mathfrak{J}$. The index i in relation $h_i(x)$, is calculated from the formula $i=\mu+\kappa+1$; in the case of minimal values $\mu=1,\kappa=0$ we have $i=2$. The maximal value of i is $i=2\mathfrak{M}=2^{\mathfrak{J}+1}$.
It is also assumed that the value of $i=1$ corresponds to the scale function $h_1\equiv 1$ in the interval $[0,1]$, which is called the characteristic function and is defined as follows:

$$\mathfrak{H}_i(x)=\begin{cases}1, & \text{for}\quad x\in[-1,1),\\ 0, & \text{otherwise.}\end{cases}$$

With the assumption of integration of Haar's wavelets in relation $\mathfrak{h}_i(x)$, we have

$$\mathfrak{p}_i(x)=\int_0^x\mathfrak{h}_i(x)\,dx,\qquad \mathfrak{q}_i(x)=\int_0^x\mathfrak{p}_i(x)\,dx.$$

It is necessary to evaluate these functions for a prescribed $\mathfrak{J}$ only once and save them in the storage of the computer.
It is convenient to pass to the matrix formulation. For this purpose, the interval $t\in[0,1]$ is divided into $\mathfrak{M}$ parts of equal length

$$\Delta t=\frac{L}{\mathfrak{M}}=\frac{B-A}{\mathfrak{M}}=\frac{1}{\mathfrak{M}},$$

where $\mathfrak{M} = 2^{\mathfrak{J}}$ and the collocation points are defined as

$$x_i = t_l = \frac{l - 0.5}{\mathfrak{M}}; \quad i = j = l = 1, 2, \ldots, \mathfrak{M}.$$

Therefore the Haar coefficient matrix $\mathfrak{H}$ is defined as $\mathfrak{H}(i, l) = \mathfrak{h}_i(t_l)$.

Next the operational matrix of integration $\mathfrak{P}$, which is a $\mathfrak{M}$ square matrix, is defined by the equation

$$(\mathfrak{P}\mathfrak{H})_{il} = \int_0^{t_l} \mathfrak{h}_i\left(t_l\right) dt,$$

and the integral operator $\mathfrak{Q}$ is introduced

$$(\mathfrak{Q}\mathfrak{H})_{il} = \int_0^{t_l} \int_0^{t} \mathfrak{h}_i\left(\tau\right) d\tau dt.$$

Chen and Hsiao defined the integral matrix in a different way [2–4]. They introduced the row vector

$$\mathfrak{h}_{(\gamma)}(t) = [\mathfrak{h}_1(t), \mathfrak{h}_2(t), \cdots, \mathfrak{h}_\gamma(t)]$$

and showed that to calculate such a matrix of order γ, the following recursive matrix equation formula holds,

$$\mathfrak{P}_\gamma = \begin{bmatrix} \mathfrak{P}_{0.5\gamma} & -\frac{1}{2\gamma}\mathfrak{H}_{0.5\gamma} \\ \frac{1}{2\gamma}\mathfrak{H}_{0.5\gamma} & 0 \end{bmatrix} = \frac{1}{2\gamma}\begin{bmatrix} 2\gamma\mathfrak{P}_{0.5\gamma} & -\mathfrak{H}_{0.5\gamma} \\ \mathfrak{H}_{0.5\gamma} & 0 \end{bmatrix},$$

whereas $\mathfrak{P}_{(1\times1)} = 0.5$.

The square matrix $\mathfrak{P}_{(\gamma\times\gamma)}$ is called the operational matrix of integration.

Theorem 1 *The integration operational matrix of fractional integration is obtained by:*

$$\begin{aligned} \mathfrak{p}_{(\gamma,i)}(t) &= \int_A^t \int_A^t \cdots \int_A^t \mathfrak{h}_i(x) dx^\gamma = \frac{1}{\Gamma(\gamma)} \int_A^t (t - x)^{\gamma-1} \mathfrak{h}_i(x) dx \\ &= \frac{1}{(\gamma - 1)!} \int_A^t (t - x)^{\gamma-1} \mathfrak{h}_i(x) dx; \quad \gamma = 1, 2, \ldots, n, \quad i = 1, 2, \ldots, \mathfrak{M}. \end{aligned}$$

Therefore for $i > 1$ implies that

$$\mathfrak{P}_{(\gamma,i)}(t) = \begin{cases} 0, & \forall t \in [A, z_1(i)], \\ \frac{1}{\gamma!}\big(t - z_1(i)\big)^\gamma, & \forall t \in [z_1(i), z_2(i)], \\ \dfrac{1}{\gamma!}\Big\{\big(t - z_1(i)\big)^\gamma - 2\big(t - z_2(i)\big)^\gamma\Big\}, & \forall t \in [z_2(i), z_3(i)], \\ \dfrac{1}{\gamma!}\Big\{\big(t - z_1(i)\big)^\gamma - 2\big(t - z_2(i)\big)^\gamma + \big(t - z_3(i)\big)^\gamma\Big\}, & \forall t \in [z_3(i), B]. \end{cases}$$

For $i = 1$, $z_1 = A$ and $z_2 = z_3 = B$, we have

$$\mathfrak{P}_{(\gamma,1)}(t) = \frac{(t - A)^\gamma}{\gamma!} = \frac{1}{\gamma!}(t - A)^\gamma.$$

3 Segmentation Method

In this section the ODE

$$\frac{d^2\mathfrak{u}}{dt^2} = \mathcal{F}\left(t, \mathfrak{u}, \frac{d\mathfrak{u}}{dt}\right);\ t\epsilon\,[0,T]\,, \tag{5}$$

is considered [1]. Instead of (5) the system of first-order equations

$$\frac{d\mathfrak{u}}{dt} = \mathfrak{v},\ \frac{d\mathfrak{v}}{dt} = \mathfrak{F}\,(t, \mathfrak{u}, \mathfrak{v})\,, \tag{6}$$

can be solved. Let us divide the time interval into n segments; the length of the i-th segment is denoted by

$$t_{i+1} = t_i + \delta t;\ i = 1, 2, \ldots, n,\quad \delta t = \mathfrak{d}_n,$$

with $(t_1 = 0,\ t_{n+1} = 1)$. Consider the i-th segment, it is convenient to introduce the local time

$$\tau = \frac{t - t_i}{\delta t},$$

and choose $\mathfrak{M}$ collocation points

$$\tau_s = \frac{1}{\mathfrak{M}}\left(s - \frac{1}{2}\right);\quad s = 1, 2, \ldots, \mathfrak{M}.$$

System (6) in local time obtains the form (dots denote differentiation with respect to τ),

$$\dot{\mathfrak{u}} = \delta t.\mathfrak{v},\quad \dot{\mathfrak{v}} = \delta t.\mathfrak{F}\,(t_i + \delta t.\tau, \mathfrak{u}, \mathfrak{v})\,. \tag{7}$$

In following it is expedient to consider the variables $\mathfrak{u}, \mathfrak{v}$ as row vectors with the components

$$\mathfrak{u}_j = \mathfrak{u}\,(\tau_j)\,,\quad \mathfrak{v}_j = \mathfrak{v}\,(\tau_j)\,;\quad j = 1, 2, \ldots, \mathfrak{M}.$$

Following Chen and Hsiao [1,2] the solution is sought in the form

$$\begin{cases} \dot{\mathfrak{u}} = a\mathfrak{H},\ \mathfrak{u} = a\mathfrak{P}\mathfrak{H} + \mathfrak{u}_i\mathfrak{E}, \\ \dot{\mathfrak{v}} = b\mathfrak{H},\ \mathfrak{v} = b\mathfrak{P}\mathfrak{H} + \mathfrak{v}_i\mathfrak{E}. \end{cases} \tag{8}$$

Where a and b are $\mathfrak{M}$ dimensional row vectors, $\mathfrak{E}$ a unit vector of the same dimension; symbols $\mathfrak{u}_i,\ \mathfrak{v}_i$ denote the values of $\mathfrak{u}$ and $\mathfrak{v}$ at the boundary $t = t_i$. Substitution of (8) into Equation (7) gives

$$\begin{cases} \dot{\mathfrak{u}} = a\mathfrak{H} = \delta t\,(b\mathfrak{P}\mathfrak{H} + \mathfrak{v}_i\mathfrak{E})\,, \\ \dot{\mathfrak{v}} = b\mathfrak{H} = \delta t.\mathfrak{F}\,(t_1, \mathfrak{u}, \mathfrak{v}) = \delta t\mathfrak{F}(t_i + \delta t.\tau, a\mathfrak{P}\mathfrak{H} + \mathfrak{u}_i E, b\mathfrak{P}\mathfrak{H} + \mathfrak{v}_i\mathfrak{E}). \end{cases} \tag{9}$$

From these matrix equations the vectors a and b are calculated, after that $\mathfrak{u}$ and $\mathfrak{v}$ can be evaluated from (4). To proceed to the next segment the values $\mathfrak{u}_{i+1},\ \mathfrak{v}_{i+1}$, must be computed. According to (8)

$$\mathfrak{u}_{i+1} = a(\mathfrak{P}\mathfrak{H})_{\tau=1} + \mathfrak{u}_i\mathfrak{E},\ \mathfrak{v}_{i+1} = b(\mathfrak{P}\mathfrak{H})_{\tau=1} + \mathfrak{v}_i\mathfrak{E}.$$

It follows from $(\mathfrak{P}\mathfrak{H})_{il} = \int_0^{t_l} \mathfrak{h}_i(t_l)dt$ that

$$(\mathfrak{P}\mathfrak{H})_{\tau=1} = \mathfrak{Y}_{\mathfrak{M}} = [1, 0, 0, \ldots, 0]$$

So, we have ($\mathfrak{a}_1, \mathfrak{b}_1$ are the first components of the vectors a, b)

$$\mathfrak{u}_{i+1} = \mathfrak{a}_1 + \mathfrak{u}_i \mathfrak{E}, \;\; \mathfrak{v}_{i+1} = \mathfrak{b}_1 + \mathfrak{v}_i \mathfrak{E} \tag{10}$$

These are the starting values for the $(i+1)-th$ segment. Note that CHM and PCA methods are special cases of Haar Wavelet Segmentation Method ($HWSM$). Also, the CHM and PCA methods are special cases of the segmentation method (SM).
Now assuming $n = 1, \; \delta t = T$ we have CHM method and in PCA method, we have one collocation point in each piece or segment, that is: $M = 1$.)
This fact allows us to combine these three methods (for example, if there are sudden changes in an area, then the segmentation method can be preferred, while in smooth areas, to achieve the necessary accuracy, PCA or CHM method will be suitable). Calculations were carried out for $PCA, HS2, HS4$ (2 or 4 collocation points in each interval) and CHM. The fastest is PCA method.

4 Nonlinear ODE Solutions

The aim of this Section is to demonstrate how the Haar wavelets can be applied for solving nonlinear differential equations. For a test problem the Non-Linear oscillator equation

$$p_0 \frac{d^2u}{dt^2} + \frac{q_0 u^m}{(1+u^2)^n} = s_0 r_e\,(t) \tag{11}$$

with the initial conditions

$$(\frac{du}{dt})_{t=0} = u_0' \; , u\,(0) = u_0$$

Now, at the beginning, by multiplying all the terms of the above differential equation by $[\delta t]^2$, we have:

$$p_0\,(\delta t)^2 \frac{d^2u}{dt^2} + \frac{(\delta t)^2\, q_0 u^m}{(1+u^2)^n} = s_0\,(\delta t)^2\, r_e\,(t)$$

The interval $t \in \; [0, T]$, is divided into n segments of the length δt and the local time $\tau = \frac{t-t_i}{\delta t}$ is introduced. Denoting

$$p = p_0, q = (\delta t)^2\, q_0 \; , \; s = s_0\,(\delta t)^2$$

Equation (11) can be rewritten in the form

$$F = p\ddot{u} + \frac{q u^m}{(1+u^2)^n} = s r_e\,(t) \tag{12}$$

Here we see that $r_e(t)$ is actually a M-dimensional vector as following:

$$r_e\left(t_i+\delta t.\tau_j\right)\ ;\ j=1,...,M$$

Solution of (12) is sought in the form

$$\begin{cases}\ddot{u}=(\delta t)^2\frac{d^2u}{dt^2}=\dot{v}=aH\ ,\\ \dot{u}=\delta t\frac{du}{dt}=v=aPH+\dot{u}_iE=v=aPH+v_iE,\\ u=aQH+\tau v_i+u_iE\end{cases} \tag{13}$$

Where H_M is Haar coefficient matrix and P,Q are $M\times M$ matrices with the following elements:

$$H\left(i,j\right)=h_i\left(t_j\right),\ P\left(i,j\right)=p_i\left(t_j\right),\ Q(i,j)=q_i(t_j)$$

If we substitute (13) into (12) the quantity F can be considered as a function of the vector a. The functional matrix is

$$F=p\ddot{u}+\frac{qu^m}{\left(1+u^2\right)^n}=paH+\frac{q\left(aQH+\tau v_i+u_iE\right)^m}{\left(E+\left(aQH+\tau v_i+u_iE\right)^2\right)^n}-s.r_e\left(\delta t.\tau+\ t_i\right)=0$$

And

$$\frac{\partial F}{\partial a}=p\ddot{u}_a+\frac{mqu_au^{m-1}\left(1+u^2\right)^n-2nu_au\left(1+u^2\right)^{n-1}qu^m}{\left(1+u^2\right)^{2n}}$$
$$=pH+\frac{mqQH\left(aQH+\tau v_i+u_iE\right)^{m-1}\left(I_M+\left(aQH+\tau v_i+u_iE\right)^2\right)^n-2nQHU\left(I_M+U^2\right)^{n-1}q\left(aQH+\tau v_i+u_iE\right)^m}{\left(I_M+\left(aQH+\tau v_i+u_iE\right)^2\right)^{2n}}$$

Or

$$\frac{\partial F}{\partial a}=pH+\frac{mqQHU^{m-1}\left(I_M+U^2\right)^n-2qnQHU\left(I_M+U^2\right)^{n-1}U^m}{\left(I_M+U^2\right)^{2n}}$$

Where U^2 denotes the element-by-element multiplication of $u(j).u(j)$.
Also the I_M is matrix of Identity and the vectors E and $r_e(t)$ are also M dimensional vectors with dimensions 1 and $r_e\left(t_i+\delta t.\tau_j\right)\ ;j=1,...,M$, respectively.

Now, choosing a starting value $a=a_0$ the corrected value in each step can be found by the Newton method

$$a_{s+1}=a_s-\frac{F(a_s)}{F'(a_s)}$$

This procedure will be repeated until necessary exactness of achieved. After that u and v are calculated from (13). The quantities $(u_{i+1}\ ,\ v_{i+1})$ will be evaluated from (10), after that transition to the next segment can be realized.

5 Numerical Examples

In this section, Three Numerical examples are given to demonstrate the accuracy and efficiency of the proposed method.

Example 1 Consider the Nonlinear Oscillator equation as follows (see [11]):

$$p_0 \frac{d^2 f}{dt^2} + \frac{q_0 f^3}{f^2 + 1} = 0 \tag{14}$$

with the following initial conditions

$$f(0) = 0 \ , \ f'(0) = 0.3$$

such that

$$p_0 = q_0 = 1 \, , \, T = 60$$

The comparison between the solution of Haar wavelet segmentation method with the Runge-Kutta solution of this problem $\forall t \in [0, 60]$ is given in table 1 and figure 2. The numerical results of this comparison convince us that method s of Haar wavelets is almost in perfect agreement with the exact or Runge-Kutta solution of this problem.

Table 1 Absolute Comparison of numerical results between Runge-Kutta solutions and Haar wavelet method for the nonlinear oscillator Equation (14) from Example 1

$n, \ \mathfrak{J}$	*Method*	$f_{ex}(60)$	$f_{Haar}(60)$	δ_n	$\sigma_n = \frac{\Vert\delta_n\Vert_2}{n}$
$n = 13500, \ \mathfrak{J} = 2$	$HS4$	-0.644993701	-0.6449978188	$9.9927e-06$	$3.3073e-07$
$n = 31500, \ \mathfrak{J} = 1$	$HS2$	-0.644993701	-0.6450018922	$1.9581e-05$	$4.2074e-07$
$n = 51500, \ \mathfrak{J} = 1$	$HS2$	-0.644993701	-0.6450018445	$1.9480e-05$	$3.2635e-07$
$n = 81500, \ \mathfrak{J} = 0$	PCA	-0.644993701	-0.6450025052	$2.1038e-05$	$2.8033e-07$
$n = 113500, \ \mathfrak{J} = 0$	PCA	-0.644993701	-0.6450024221	$2.0865e-05$	$2.3498e-07$

Example 2 Consider the Nonlinear Oscillator equation as follows (see [11]):

$$\theta'' + p_0 \sin\theta = 0 \tag{15}$$

with the following initial conditions

$$\theta(0) = \frac{\pi}{3} \ , \ \theta'(0) = 0$$

such that

$$p_0 = \frac{g}{L} \ ; L = 1m \, , \, T = 10$$

Now, at the beginning, by multiplying all the terms of the above differential equation by $[\delta t]^2$, we have:

$$(\delta t)^2 \theta''(t) + (\delta t)^2 p_0 \sin\ \theta = 0$$

Equation (15) can be rewritten in the form

$$F = \ddot{\theta} + p \sin\ \theta = 0 \tag{16}$$

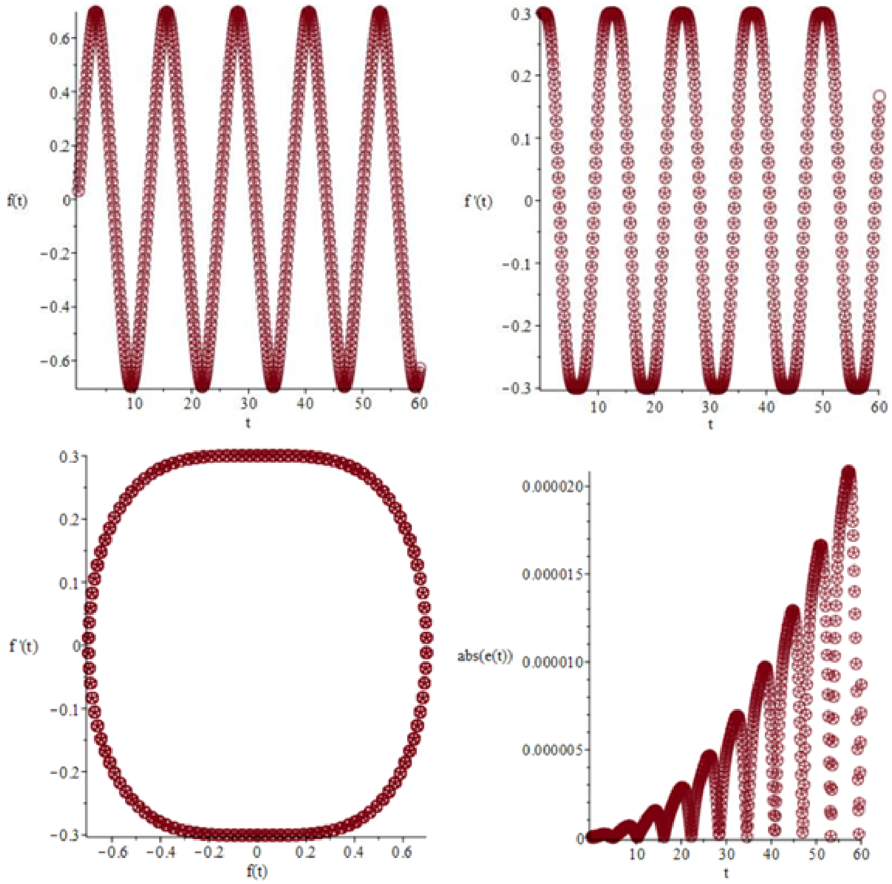

Fig. 2 Comparison between the Runge-Kutta and numerical solutions using Haar wavelet for each $t \in [0, T] = [0, 60]$ for nonlinear oscillator Equation (14) with $n = 31500$, $\mathfrak{J} = 2$ from Example 1.

Solution of (15) is sought in the form

$$\begin{cases} \ddot{\theta} = aH \ , \\ \dot{\theta} = \varphi = aPH + \dot{\theta}_i E = aPH + \varphi_i E, \\ \theta = aQH + \tau\varphi_i + \theta_i E \end{cases} \tag{17}$$

If we substitute (17) into (16) the quantity F can be considered as a function of the vector a. The functional matrix is

$$F = aH + p \sin \ (aQH + \tau\varphi_i + \theta_i E)$$

and consequently

$$\frac{\partial F}{\partial a} = \ddot{\theta}_a + p\theta_a \cos\theta = H + pQH \cos\left(aQH + \tau\varphi_i + \theta_i E\right)$$

The comparison between the solution of Haar wavelet segmentation method with the Runge-Kutta solution of this problem $\forall t \in [0, 10]$ is given in table 2 and figure 3. The numerical results of this comparison convince us that method s of Haar wavelets is almost in perfect agreement with the exact or Runge-Kutta solution of this problem.

Table 2 Absolute Comparison of numerical results between Runge-Kutta solutions and Haar wavelet method for nonlinear oscillator Equation (15) from Example 2

$n,\ \mathfrak{J}$	$Method$	$f_{ex}(10)$	$f_{Haar}(10)$	δ_n	$\sigma_n = \frac{\|\|\delta_n\|\|_2}{n}$
$n = 13500,\ \mathfrak{J} = 2$	$HS4$	-0.65187083	-0.6518634227	$7.6910e-06$	$2.2013e-07$
$n = 31500,\ \mathfrak{J} = 1$	$HS2$	-0.65187083	-0.6518631376	$7.8807e-06$	$1.4612e-07$
$n = 51500,\ \mathfrak{J} = 1$	$HS2$	-0.65187083	-0.6518630636	$7.9390e-06$	$1.1483e-07$
$n = 81500,\ \mathfrak{J} = 0$	PCA	-0.65187083	-0.6518628828	$8.0685e-06$	$9.2697e-08$
$n = 113500,\ \mathfrak{J} = 0$	PCA	-0.65187083	-0.6518630326	$7.8853e-06$	$7.6527e-08$

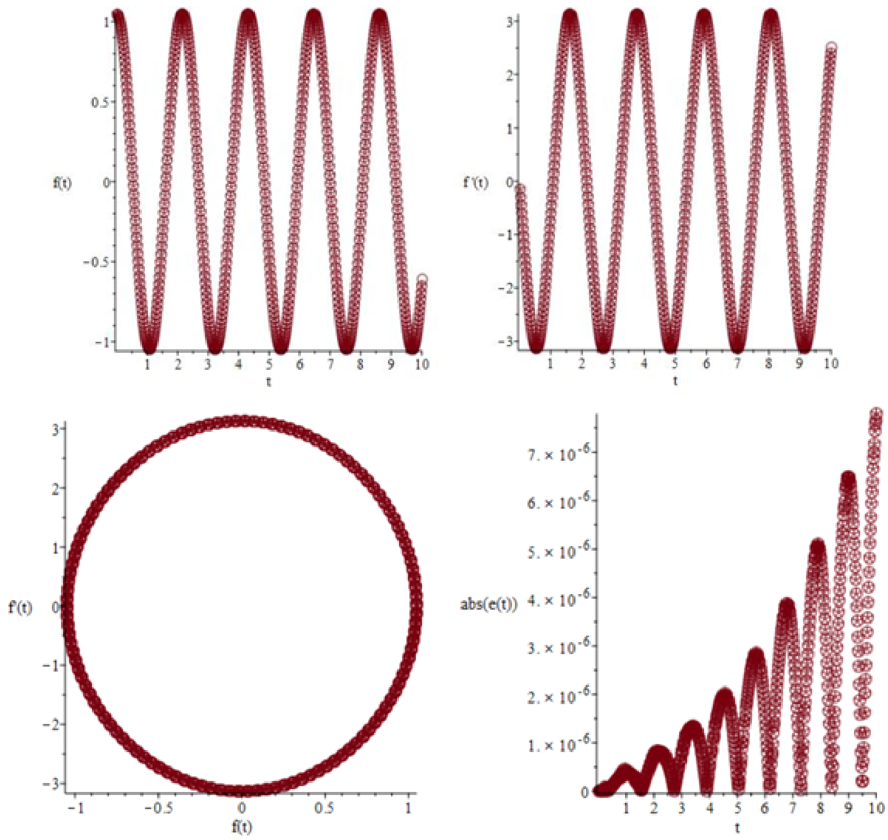

Fig. 3 Comparison between the Runge-Kutta and numerical solutions using Haar wavelet for each $t \in [0, T] = [0, 10]$ for nonlinear oscillator Equation (15) with $n = 31500$, $\mathfrak{J} = 2$ from Example 2.

6 Conclusion

In this paper, the Haar Wavelet Segmentation Method, was used to solve the fluctuating nonlinear differential equations that have many applications in various branches of physics and engineering, and other basic and applied sciences, and the high accuracy of this method was revealed in solving such equations.

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