Global Analysis and Discrete Mathematics
Volume x, Issue x, pp. xx–xx
ISSN: 2476-5341

Upwind Implicit Scheme for the Numerical Solution of Stochastic Advection–Diffusion Partial Differential Equations

Mehran Namjoo · Mehran Aminian · Ali Mohebbian · Mehdi Karami · Hossein Salmei

Received: 10 August 2024 / Accepted: 23 September 2024

Abstract Stochastic partial differential equations (SPDEs) are significant in various fields such as epidemiology, mechanics, microelectronics, chemistry, and finance. Obtaining analytical solutions for SPDEs is either difficult or impossible; therefore, researchers are very interested in effective numerical methods for studying the behavior of these equations. In this paper, we introduce a stochastic finite difference (SFD) scheme for the numerical solution of the Itô stochastic advection–diffusion equation. We discuss the consistency, stability, and convergence of the scheme, and we also determine its order of convergence. Finally, to validate the effectiveness and accuracy of the SFD scheme, we analyze the numerical results and compare them with those from existing SFD schemes.

Keywords Itô stochastic partial differential equation · Finite difference · Consistency · Stability · Convergence

Mathematics Subject Classification (2010) 60H15 · 65M12

M. Namjoo (Corresponding Author)
Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran.
E-mail: namjoo@vru.ac.ir

M. Aminian
Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran.
E-mail: mehran.aminian@vru.ac.ir

A. Mohebbian
Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran.
E-mail: a.mohebbian@stu.vru.ac.ir

M. Karami
Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran.
E-mail: m.karami@vru.ac.ir

H. Salmei
Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran.
E-mail: salmei@vru.ac.ir

1 Introduction

Over the past several decades, SPDEs have been utilized to model realistic phenomena because they accurately represent real behaviors. The footprints of SPDEs can be seen in many fields, including nonlinear filtering [1], turbulent flows [2], population biology [3, 5], microscopic particle dynamics [4], groundwater flow [15], plasma physics, and finance. Only a limited number of SPDEs can be solved using analytical techniques, while most cannot be addressed with these methods [16]. Consequently, several numerical methods have been developed to solve these equations [17–19]. Many researchers have investigated numerical approaches for approximating the solutions of SPDEs. In [6], Allen et al. applied finite element methods to solve SPDEs. In [7–10], Namjoo et al. utilized SFD schemes for solving SPDEs. In [12], a stochastic compact finite difference scheme was proposed for solving a class of SPDEs. In [13], Roth applied finite difference schemes to solve SPDEs. In [20], high–resolution finite volume methods were employed to address the numerical solution of SPDEs. Furthermore, in [21], a stochastic compact finite difference scheme was used to solve a stochastic fractional partial differential equation. Fractional calculus, as a generalization of classical calculus, is a powerful mathematical framework for representing and analyzing physical systems with complex dynamics that cannot be well described using standard integer–order models. Recent studies have shown that fractional–order differential equations can represent complex dynamic features more accurately than ordinary differential equations. This is because fractional–order derivatives and integrals can effectively describe the characteristic of memory effects, which are a crucial element in many real–world phenomena. In recent years, the use of fractional–order derivatives has significantly increased, and they have been widely applied in modeling real–world phenomena, as well as in exploring disease transmission and control processes [23, 25, 27]. Furthermore, numerous studies have been conducted in recent years on biological models and the generalized Schrodinger equation that incorporate fractional–order derivatives [24, 26, 28]. This paper introduces an efficient SFD scheme for solving stochastic advection–diffusion equations.

The reminder of this paper is organized as follows: In Section 2, we construct a reliable implicit SFD scheme to implement numerical solutions for stochastic advection–diffusion equations. In Section 3, we study the consistency, stability, and convergence properties of the proposed SFD scheme. Section 4 presents some numerical simulations, and finally, the paper concludes with remarks in the last section.

2 The upwind scheme for SPDEs

In this section, we focus on constructing an efficient numerical method for approximating the solution of the stochastic advection–diffusion equation as, described in [8].

$$U_t(x,t) + \nu U_x(x,t) = \gamma U_{xx}(x,t) + \sigma U(x,t)\dot{W}(t),\ (x,t) \in [0,1] \times [0,1], \quad (1)$$

with initial and boundary conditions

$$U(x,0) = U_0(x), \quad U(0,t) = U_1(t), \quad U(1,t) = U_2(t), \quad x \in [0,1], \quad t \in [0,1].$$

The function $U(x,t)$ represents the concentration, while the parameter ν denotes the viscosity of material flow or convection. The constant γ describes the rate of diffusion or dispersion of the material in space, and the coefficient σ characterizes the intensity of random fluctuations. Moreover, $W(t)$ is a Wiener process, and $\dot{W}(t)$ is referred to as a white noise process [11]. Equation (1) can be used to model the spread of pollution in air or soil, particularly under conditions where convection and random fluctuations (such as wind or rain) affect the distribution of pollutants. It can also be applied to heat transfer processes to model temperature distribution in environments influenced by convection and randomness. In chemical reactions involving convection and the diffusion of substances, it effectively describes concentration dynamics. Additionally, it can be utilized to model asset prices and financial processes, explaining price fluctuations influenced by random and convective factors, such as market trends. In plasma physics, it describes the movement of particles in electric and magnetic fields affected by convection and random fluctuations. In ecology, it is useful for modeling the spread of biological species and their environmental impacts. Lastly, in biology, it can model the diffusion of nutrients or drugs in biological tissues, influenced by blood flow and random variations.

Let us consider a uniform time–space lattice with the step sizes Δx and Δt. Suppose U_k^n represents the approximate solution at the nodal point $x_k = k\Delta x$ and $t_n = n\Delta t$, where $\Delta x = x_{k+1} - x_k$ and $\Delta t = t_{n+1} - t_n$, for $0 \le k \le M-1$ and $0 \le n \le N-1$. To construct an SFD scheme for the SPDE (1), the time and space partial derivatives can be approximated as follows [14,22]

$$\begin{aligned} U_t(k\Delta x, n\Delta t) &\approx \frac{U_k^{n+1} - U_k^n}{\Delta t}, \\ U_x(k\Delta x, n\Delta t) &\approx \frac{U_k^{n+1} - U_{k-1}^{n+1}}{\Delta x}, \\ U_{xx}(k\Delta x, n\Delta t) &\approx \frac{U_{k+1}^{n+1} - 2U_k^{n+1} + U_{k-1}^{n+1}}{\Delta x^2}. \end{aligned} \tag{2}$$

By substituting the approximations from (2) into (1) and using the approximation of the white noise process for the problem (1), we obtain the stochastic upwind implicit scheme as follows:

$$-(\nu\lambda + \gamma\rho)U_{k-1}^{n+1} + (1 + \nu\lambda + 2\gamma\rho)U_k^{n+1} - \gamma\rho U_{k+1}^{n+1} = U_k^n + \sigma U_k^n \Delta W_n, \tag{3}$$

where $\lambda = \frac{\Delta t}{\Delta x}$, $\rho = \frac{\Delta t}{\Delta x^2}$, and $\Delta W_n = W((n+1)\Delta t) - W(n\Delta t)$ is a Gaussian distribution with zero mean and variance Δt [11].

In continuation, we examine several aspects of the stochastic scheme (3). To achieve this, we integrate both sides of the SPDE (1) with respect to time over the interval $[0, t]$, resulting in:

$$U(x,t) - U(x,0) + \nu \int_0^t U_x(x,s)\mathrm{d}s = \gamma \int_0^t U_{xx}(x,s)\mathrm{d}s + \sigma \int_0^t U(x,s)\mathrm{d}W(s). \tag{4}$$

By substituting the values $t = t_{n+1}$ and t_n into (4), one obtains:

$$\begin{aligned} U(x,t_{n+1}) - U(x,0) + \nu \int_0^{t_{n+1}} U_x(x,s)\mathrm{d}s =& \gamma \int_0^{t_{n+1}} U_{xx}(x,s)\mathrm{d}s \\ &+ \sigma \int_0^{t_{n+1}} U(x,s)\mathrm{d}W(s), \end{aligned} \tag{5}$$

and

$$\begin{aligned} U(x,t_n) - U(x,0) + \nu \int_0^{t_n} U_x(x,s)\mathrm{d}s =& \gamma \int_0^{t_n} U_{xx}(x,s)\mathrm{d}s \\ &+ \sigma \int_0^{t_n} U(x,s)\mathrm{d}W(s). \end{aligned} \tag{6}$$

Subtracting (6) from (5) and setting $x = x_n$, one concludes that:

$$\begin{aligned} U(x_k,t_{n+1}) - U(x_k,t_n) + \nu \int_{t_n}^{t_{n+1}} U_x(x_k,s)\mathrm{d}s - \gamma \int_{t_n}^{t_{n+1}} U_{xx}(x_k,s)\mathrm{d}s \\ - \sigma \int_{t_n}^{t_{n+1}} U(x_k,s)\mathrm{d}W(s) = 0. \end{aligned} \tag{7}$$

The equation (7) can be considered as:

$$\mathcal{L}U(x_k,t_n) = \mathcal{F},$$

in a way that

$$\begin{aligned} \mathcal{L}U(x_k,t_n) =& U(x_k,t_{n+1}) - U(x_k,t_n) + \nu \int_{t_n}^{t_{n+1}} U_x(x_k,s)\,\mathrm{d}s \\ &- \gamma \int_{t_n}^{t_{n+1}} U_{xx}(x_k,s)\,\mathrm{d}s - \sigma \int_{t_n}^{t_{n+1}} U(x_k,s)\,\mathrm{d}W(s), \end{aligned} \tag{8}$$

and $\mathcal{F} = 0$. To obtain the difference operator associated with the stochastic difference scheme (3), we consider the following approximations:

$$\begin{aligned} \int_{t_n}^{t_{n+1}} U_x(x_k,s)\,\mathrm{d}s &\approx \int_{t_n}^{t_{n+1}} U_x(x_k,t_n)\,\mathrm{d}s, \\ \int_{t_n}^{t_{n+1}} U_{xx}(x_k,s)\,\mathrm{d}s &\approx \int_{t_n}^{t_{n+1}} U_{xx}(x_k,t_n)\,\mathrm{d}s, \\ \int_{t_n}^{t_{n+1}} U(x_k,s)\,\mathrm{d}W(s) &\approx \int_{t_n}^{t_{n+1}} U(x_k,t_n)\,\mathrm{d}W(s). \end{aligned} \tag{9}$$

By replacing the approximations in (2) with those in (9) and using (7), one achieves:

$$U(x_k,t_{n+1})-U(x_k,t_n)+\frac{\nu\Delta t}{\Delta x}(U(x_k,t_{n+1})-U(x_{k-1},t_{n+1}))$$
$$-\frac{\gamma\Delta t}{\Delta x^2}\big(U(x_{k+1},t_{n+1})-2U(x_k,t_{n+1})+U(x_{k-1},t_{n+1})\big)-\sigma U(x_k,t_n)\Delta W_n$$
$$=0. \tag{10}$$

The stochastic difference scheme (10) can be expressed in the following manner:

$$\mathcal{L}_k^n U_k^n=\mathcal{F}_k^n,$$

where

$$\mathcal{L}_k^n U_k^n =U(x_k,t_{n+1})-U(x_k,t_n)+\frac{\nu\Delta t}{\Delta x}(U(x_k,t_{n+1})-U(x_{k-1},t_{n+1}))$$
$$-\frac{\gamma\Delta t}{\Delta x^2}\big(U(x_{k+1},t_{n+1})-2U(x_k,t_{n+1})+U(x_{k-1},t_{n+1})\big)$$
$$-\sigma U(x_k,t_n)\Delta W_n, \tag{11}$$

and $\mathcal{F}_k^n=0$. Consider an SPDE of the form $\mathcal{L}U=\mathcal{F}$. Let $\mathcal{L}_k^n U_k^n=\mathcal{F}_k^n$ represent the proposed difference scheme. To investigate consistency, stability, and convergence, it is necessary to consider a norm. To this end, let $\{U_k^n\}$ be a sequence of numerical approximations obtained from the stochastic difference scheme (3). Define

$$\|U^n\|=\sqrt{\sup_{0\le k\le M}|U_k^n|^2},$$

where $U^n=(U_0^n,U_1^n,\dots,U_M^n)$. For further details on the concepts of consistency, stability, and convergence, see [13].

Definition 1 A stochastic difference scheme $\mathcal{L}_k^n U_k^n=\mathcal{F}_k^n$ is said to be consistent in mean square with the SPDE $\mathcal{L}U=\mathcal{F}$ at the point $(x,t)\in[0,1]\times[0,1]$ if, for any continuously differentiable function $\Upsilon(x,t)$, we have:

$$\mathbb{E}\|\left(\mathcal{L}\Upsilon(k\Delta x,n\Delta t)-\mathcal{F}(k\Delta x,n\Delta t)\right)-\left(\mathcal{L}_k^n\Upsilon(k\Delta x,n\Delta t)-\mathcal{F}_k^n\right)\|^2\to 0,$$

as $\Delta x\to 0$, $\Delta t\to 0$ and $(k\Delta x,(n+1)\Delta t)\to(x,t)$.

Definition 2 The stochastic difference scheme $\mathcal{L}_k^n u_k^n=\mathcal{F}_k^n$, which approximates the SPDE $\mathcal{L}v=\mathcal{F}$, is said to be convergent in mean square at time $t=(n+1)\Delta t$ if $\mathbb{E}\|u^{n+1}-v^{n+1}\|^2\to 0$ as $\Delta x\to 0$ and $\Delta t\to 0$.

Definition 3 Let Ω be a domain in $\mathbb{R}^n$ $(n\ge 2)$. The Sobolev space of order m, denoted as $H^m(\Omega)$, is defined as the set of functions in $L^2(\Omega)$ for which all weak partial derivatives, up to and including those of order m, are also in $L^2(\Omega)$.

3 Consistency, stability, and convergence of the stochastic upwind scheme

In this section, we demonstrate the properties of consistency, stability, and convergence of the stochastic upwind scheme (3).

Theorem 1 *The stochastic difference scheme* (3) *is consistent in mean square in the sense of Definition 1.*

Proof Suppose that $\Upsilon(x,t)$ is a smooth function. It follows from equations (8) and (11) that:

$$\begin{aligned}\mathcal{L}(\Upsilon(k\Delta x,n\Delta t)) =& \Upsilon(k\Delta x,(n+1)\Delta t)-\Upsilon(k\Delta x,n\Delta t)\\ &+\nu\int_{n\Delta t}^{(n+1)\Delta t}\Upsilon_x(k\Delta x,s)\,\mathrm{d}s-\gamma\int_{n\Delta t}^{(n+1)\Delta t}\Upsilon_{xx}(k\Delta x,s)\,\mathrm{d}s\\ &-\sigma\int_{n\Delta t}^{(n+1)\Delta t}\Upsilon(k\Delta x,s)\,\mathrm{d}W(s),\end{aligned}$$

and

$$\begin{aligned}&\mathcal{L}_k^n\Upsilon(k\Delta x,n\Delta t)=\Upsilon(k\Delta x,(n+1)\Delta t)-\Upsilon(k\Delta x,n\Delta t)\\ &+\nu\Delta t\frac{\Upsilon(k\Delta x,(n+1)\Delta t)-\Upsilon((k-1)\Delta x,(n+1)\Delta t)}{\Delta x}\\ &-\gamma\Delta t\frac{\Upsilon((k+1)\Delta x,(n+1)\Delta t)-2\Upsilon(k\Delta x,(n+1)\Delta t)+\Upsilon((k-1)\Delta x,(n+1)\Delta t)}{\Delta x^2}\\ &-\sigma\Upsilon(k\Delta x,n\Delta t)(W((n+1)\Delta t)-W(n\Delta t)).\end{aligned}$$

Using the square property of the Itô integral [11], one obtains:

$$\begin{aligned}&\mathbb{E}|\mathcal{L}(\Upsilon(k\Delta x,n\Delta t))-\mathcal{L}_k^n\Upsilon(k\Delta x,n\Delta t)|^2\\ &=\mathbb{E}\Bigg|\nu\int_{n\Delta t}^{(n+1)\Delta t}\left(\Upsilon_x(k\Delta x,s)-\frac{\Upsilon(k\Delta x,(n+1)\Delta t)-\Upsilon((k-1)\Delta x,(n+1)\Delta t)}{\Delta x}\right)\mathrm{d}s\\ &\quad-\gamma\int_{n\Delta t}^{(n+1)\Delta t}\Bigg(\Upsilon_{xx}(k\Delta x,s)\\ &\quad-\frac{\Upsilon((k+1)\Delta x,(n+1)\Delta t)-2\Upsilon(k\Delta x,(n+1)\Delta t)+\Upsilon((k-1)\Delta x,(n+1)\Delta t)}{\Delta x^2}\Bigg)\mathrm{d}s\\ &\quad-\sigma\int_{n\Delta t}^{(n+1)\Delta t}\Big(\Upsilon(k\Delta x,s)-\Upsilon(k\Delta x,n\Delta t)\Big)\mathrm{d}W(s)\Bigg|^2\\ &\le 4\nu^2\mathbb{E}\left|\int_{n\Delta t}^{(n+1)\Delta t}\left(\Upsilon_x(k\Delta x,s)-\frac{\Upsilon(k\Delta x,(n+1)\Delta t)-\Upsilon((k-1)\Delta x,(n+1)\Delta t)}{\Delta x}\right)\mathrm{d}s\right|^2\\ &\quad+4\gamma^2\mathbb{E}\Bigg|\int_{n\Delta t}^{(n+1)\Delta t}\Bigg(\Upsilon_{xx}(k\Delta x,s)\\ &\quad-\frac{\Upsilon((k+1)\Delta x,(n+1)\Delta t)-2\Upsilon(k\Delta x,(n+1)\Delta t)+\Upsilon((k-1)\Delta x,(n+1)\Delta t)}{\Delta x^2}\Bigg)\mathrm{d}s\Bigg|^2\\ &\quad+4\sigma^2\int_{n\Delta t}^{(n+1)\Delta t}|\Upsilon(k\Delta x,s)-\Upsilon(k\Delta x,n\Delta t)|^2\,\mathrm{d}s.\end{aligned}$$

Since $\Upsilon(x,t)$ is a deterministic function, $\mathbb{E}|\mathcal{L}(\Upsilon(k\Delta x, n\Delta t)) - L_k^n \Upsilon(k\Delta x, n\Delta t)|^2$ converges to zero as n, $k \to \infty$. Therefore, the stochastic upwind scheme (3) is consistent with the SPDE (1).

Let $\hat{U}^{n+1}$ be the Fourier transform of U^{n+1}. The Fourier inversion formula leads to:

$$U_m^{n+1} = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{\Delta x}}^{\frac{\pi}{\Delta x}} e^{im\Delta x \xi} \hat{U}^{n+1}(\xi)\, \mathrm{d}\xi,$$

where

$$\hat{U}^{n+1} = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\Delta x \xi} U_m^{n+1} \Delta x, \tag{12}$$

and ξ is a real variable. One can use the Von Neumann method to analyze the stability of an SFD scheme. By substituting (12) into the SFD scheme and utilizing the properties of the Fourier transformation, one attains:

$$\hat{U}^{n+1}(\xi) = g(\Delta x\xi, \Delta t, \Delta x)\hat{U}^n(\xi),$$

where $\hat{U}^{n+1}(\xi)$ is the Fourier transformation of $U^{n+1}(\xi)$. Hence,

$$\mathbb{E}|g(\Delta x\xi, \Delta t, \Delta x)|^2 \le 1 + K\Delta t,$$

is a necessary and sufficient condition for the stability of the SFD scheme [13].

The stability of the SFD scheme (3) is demonstrated in the following theorem.

Theorem 2 *The stochastic upwind scheme* (3) *is unconditionally stable according to Fourier transform analysis.*

Proof By substituting (12) into (3), one arrives at:

$$\begin{aligned} &-(\nu\lambda + \gamma\rho)e^{-i\Delta x\xi}\hat{U}^{n+1}(\xi) + (1 + \nu\lambda + 2\gamma\rho)\hat{U}^{n+1}(\xi) - \gamma\rho e^{i\Delta x\xi}\hat{U}^{n+1}(\xi) \\ &\quad = \hat{U}^n(\xi) + \sigma\hat{U}^n(\xi)(W((n+1)\Delta t) - W(n\Delta t)). \end{aligned}$$

It follows that:

$$\begin{aligned} \hat{U}^{n+1}(\xi) = \Bigg(&\frac{1}{-(\nu\lambda + \gamma\rho)e^{-i\Delta x\xi} + (1 + \nu\lambda + 2\gamma\rho) - \gamma\rho e^{i\Delta x\xi}} \\ &+ \sigma \frac{W((n+1)\Delta t) - W(n\Delta t)}{-(\nu\lambda + \gamma\rho)e^{-i\Delta x\xi} + (1 + \nu\lambda + 2\gamma\rho) - \gamma\rho e^{i\Delta x\xi}} \Bigg) \hat{U}^n(\xi). \end{aligned}$$

Thus, the amplification factor of the stochastic upwind scheme is:

$$\begin{aligned} g(\Delta x\xi, \Delta t, \Delta x) = &\frac{1}{-(\nu\lambda + \gamma\rho)e^{-i\Delta x\xi} + (1 + \nu\lambda + 2\gamma\rho) - \gamma\rho e^{i\Delta x\xi}} \\ &+ \sigma \frac{W((n+1)\Delta t) - W(n\Delta t)}{-(\nu\lambda + \gamma\rho)e^{-i\Delta x\xi} + (1 + \nu\lambda + 2\gamma\rho) - \gamma\rho e^{i\Delta x\xi}}. \end{aligned}$$

Since the increments of the Wiener process are independent, one can deduce that

$$\mathbb{E}|g(\Delta x\xi, \Delta t, \Delta x)|^2 = \left(\frac{1}{-(\nu\lambda+\gamma\rho)e^{-i\Delta x\xi} + (1+\nu\lambda+2\gamma\rho) - \gamma\rho e^{i\Delta x\xi}}\right)^2 + \left(\frac{\sigma}{-(\nu\lambda+\gamma\rho)e^{-i\Delta x\xi} + (1+\nu\lambda+2\gamma\rho) - \gamma\rho e^{i\Delta x\xi}}\right)^2 \Delta t.$$

It is obvious that

$$\left|\frac{1}{-(\nu\lambda+\gamma\rho)e^{-i\Delta x\xi} + (1+\nu\lambda+2\gamma\rho) - \gamma\rho e^{i\Delta x\xi}}\right| \leq 1.$$

Furthermore, there exists a positive constant K such that:

$$\left|\frac{\sigma}{-(\nu\lambda+\gamma\rho)e^{-i\Delta x\xi} + (1+\nu\lambda+2\gamma\rho) - \gamma\rho e^{i\Delta x\xi}}\right|^2 \leq K.$$

This demonstrates that the stochastic upwind scheme (3) is unconditionally stable.

In the reminder of the paper, we consider v^{n+1} and u^{n+1} as the exact and numerical solutions at the time level $n+1$, respectively.

Theorem 3 *Let $v \in H^4((0,1)\times(0,1))$. Then, the stochastic upwind scheme* (3) *for the SPDE* (1) *is convergent in mean square with respect to $\|.\|_\infty$.*

Proof The solution v_k^{n+1} can be expressed using the Taylor expansions $v_x(x,s)$ and $v_{xx}(x,s)$ with respect to the spatial variable as follows:

$$\begin{aligned} v_k^{n+1} &= v_k^n - \nu \int_{n\Delta t}^{(n+1)\Delta t} v_x(x,s)|_{x=x_k}\, \mathrm{d}s + \gamma \int_{n\Delta t}^{(n+1)\Delta t} v_{xx}(x,s)|_{x=x_k}\, \mathrm{d}s \\ &\quad + \sigma \int_{n\Delta t}^{(n+1)\Delta t} v(x,s)|_{x=x_k}\, \mathrm{d}W(s) \\ &= v_k^n - \nu \int_{n\Delta t}^{(n+1)\Delta t} \left(\frac{v_k^{n+1} - v_{k-1}^{n+1}}{\Delta x} - \Delta t v_{xt}(k\Delta x, s+\alpha\Delta t) \right. \\ &\quad \left. + \frac{\Delta x}{2} v_{xx}((k+\eta)\Delta x, s+\Delta t) \right) \mathrm{d}s \\ &\quad + \gamma \int_{n\Delta t}^{(n+1)\Delta t} \left(\frac{v_{k+1}^{n+1} - 2v_k^{n+1} + v_{k-1}^{n+1}}{\Delta x^2} - \frac{\Delta x^2}{4!} \Big(v_{xxxx}((k+\beta)\Delta x, s+\Delta t) \right. \\ &\quad \left. + v_{xxxx}((k+\delta)\Delta x, s+\Delta t) \Big) - \Delta t v_{xxt}(k\Delta x, s+\vartheta\Delta t) \right) \mathrm{d}s \\ &\quad + \sigma \int_{n\Delta t}^{(n+1)\Delta t} v(x,s)|_{x=x_k}\, \mathrm{d}W(s), \end{aligned}$$

where $\alpha,\ \eta,\ \beta,\ \delta,\ \vartheta \in (0,1)$. Let $z_k^n = v_k^n - u_k^n$ be the error at the nodal point (x_k, t_n). Hence, one obtains:

$$
\begin{aligned}
z_k^{n+1} = v_k^n - u_k^n - \nu \int_{n\Delta t}^{(n+1)\Delta t} \Bigg(& \frac{v_k^{n+1} - v_{k-1}^{n+1}}{\Delta x} - \frac{u_k^{n+1} - u_{k-1}^{n+1}}{\Delta x} \\
& + \nu \Delta t v_{xx}(k\Delta x, s + \alpha \Delta t) \\
& - \gamma \Delta t v_{xxx}(k\Delta x, s + \alpha\Delta t) + \frac{\Delta x}{2} v_{xx}((k+\eta)\Delta x, s + \Delta t) \Bigg) \mathrm{d}s \\
& + \gamma \int_{n\Delta t}^{(n+1)\Delta t} \Bigg(\frac{v_{k+1}^{n+1} - 2v_k^{n+1} + v_{k-1}^{n+1}}{\Delta x^2} - \frac{u_{k+1}^{n+1} - 2u_k^{n+1} + u_{k-1}^{n+1}}{\Delta x^2} \\
& - \frac{\Delta x^2}{4!} \Big(v_{xxxx}((k+\beta)\Delta x, s + \Delta t) + v_{xxxx}((k+\delta)\Delta x, s+\Delta t) \Big) \\
& + \nu \Delta t v_{xxx}(k\Delta x, s + \vartheta \Delta t) - \gamma \Delta t v_{xxxx}(k\Delta x, s + \vartheta\Delta t) \Bigg) \mathrm{d}s \\
& + \nu\sigma\Delta t \int_{n\Delta t}^{(n+1)\Delta t} v_x(x,s)|_{x=x_k} \,\mathrm{d}W(s) \\
& - \gamma\sigma\Delta t \int_{n\Delta t}^{(n+1)\Delta t} v_{xx}(x,s)|_{x=x_k} \,\mathrm{d}W(s) \\
& + \sigma \int_{n\Delta t}^{(n+1)\Delta t} (v(x,s)|_{x=x_k} - u_k^n) \,\mathrm{d}W(s).
\end{aligned}
$$

This implies that:

$$
\begin{aligned}
z_k^{n+1} = z_k^n - \nu\lambda(z_k^{n+1} - z_{k-1}^{n+1}) - \nu \int_{n\Delta t}^{(n+1)\Delta t} \Big(& \nu\Delta t v_{xx}(k\Delta x, s + \alpha\Delta t) \\
& - \gamma \Delta t v_{xxx}(k\Delta x, s+\alpha\Delta t) + \frac{\Delta x}{2} v_{xx}((k+\eta)\Delta x, s+\Delta t) \Big) \mathrm{d}s \\
& + \gamma\rho(z_{k+1}^{n+1} - 2z_k^{n+1} + z_{k-1}^{n+1}) \\
& + \gamma \int_{n\Delta t}^{(n+1)\Delta t} \Bigg(- \frac{\Delta x^2}{4!} \Big(v_{xxxx}((k+\beta)\Delta x, s+\Delta t) \\
& + v_{xxxx}((k+\delta)\Delta x, s+\Delta t) \Big) + \nu\Delta t v_{xxx}(k\Delta x, s+\vartheta\Delta t) \\
& - \gamma\Delta t v_{xxxx}(k\Delta x, s+\vartheta\Delta t) \Bigg) \mathrm{d}s \\
& + \nu\sigma\Delta t \int_{n\Delta t}^{(n+1)\Delta t} v_x(x,s)|_{x=x_k} \,\mathrm{d}W(s) \\
& - \gamma\sigma\Delta t \int_{n\Delta t}^{(n+1)\Delta t} v_{xx}(x,s)|_{x=x_k} \,\mathrm{d}W(s)
\end{aligned}
$$

$$+\,\sigma\int_{n\Delta t}^{(n+1)\Delta t}(v(x,s)|_{x=x_k}-u_k^n)\,\mathrm{d}W(s).$$

So,

$$\begin{aligned}
&(1+\nu\lambda+2\gamma\rho)z_k^{n+1}-(\nu\lambda+\gamma\rho)z_{k-1}^{n+1}-\gamma\rho z_{k+1}^{n+1}\\
&=z_k^n-\nu\int_{n\Delta t}^{(n+1)\Delta t}\Big(\nu\Delta t v_{xx}(k\Delta x,s+\alpha\Delta t)-\gamma\Delta t v_{xxx}(k\Delta x,s+\alpha\Delta t)\\
&\quad+\frac{\Delta x}{2}v_{xx}((k+\eta)\Delta x,s+\Delta t)\Big)\,\mathrm{d}s\\
&\quad+\gamma\int_{n\Delta t}^{(n+1)\Delta t}\Bigg(-\frac{\Delta x^2}{4!}\Big(v_{xxxx}((k+\beta)\Delta x,s+\Delta t)+v_{xxxx}((k+\delta)\Delta x,s+\Delta t)\Big)\\
&\quad+\nu\Delta t v_{xxx}(k\Delta x,s+\vartheta\Delta t)-\gamma\Delta t v_{xxxx}(k\Delta x,s+\vartheta\Delta t)\Bigg)\,\mathrm{d}s\\
&\quad+\nu\sigma\Delta t\int_{n\Delta t}^{(n+1)\Delta t}v_x(x,s)|_{x=x_k}\,\mathrm{d}W(s)-\gamma\sigma\Delta t\int_{n\Delta t}^{(n+1)\Delta t}v_{xx}(x,s)|_{x=x_k}\,\mathrm{d}W(s)\\
&\quad+\sigma\int_{n\Delta t}^{(n+1)\Delta t}(v(x,s)|_{x=x_k}-u_k^n)\,\mathrm{d}W(s),
\end{aligned}$$

where $\lambda=\frac{\Delta t}{\Delta x}$ and $\rho=\frac{\Delta t}{\Delta x^2}$. Applying $\mathbb{E}|.|^2$ to the above equation and using the following inequality:

$$\mathbb{E}|X+Y+Z+R+S|^2\le 4\mathbb{E}|X|^2+8\mathbb{E}|Y|^2+16\mathbb{E}|Z|^2+16\mathbb{E}|R|^2+2\mathbb{E}|S|^2,$$

we gain

$$\begin{aligned}
&\mathbb{E}\left|(1+\nu\lambda+2\gamma\rho)z_k^{n+1}-(\nu\lambda+\gamma\rho)z_{k-1}^{n+1}-\gamma\rho z_{k+1}^{n+1}\right|^2\\
&\le 4\mathbb{E}|z_k^n|^2+8\mathbb{E}\Bigg|-\nu\int_{n\Delta t}^{(n+1)\Delta t}\Big(\nu\Delta t v_{xx}(k\Delta x,s+\alpha\Delta t)-\gamma\Delta t v_{xxx}(k\Delta x,s+\alpha\Delta t)\\
&\quad+\frac{\Delta x}{2}v_{xx}((k+\eta)\Delta x,s+\Delta t)\Big)\,\mathrm{d}s+\gamma\int_{n\Delta t}^{(n+1)\Delta t}\Bigg(-\frac{\Delta x^2}{4!}\Big(v_{xxxx}((k+\beta)\Delta x,s+\Delta t)\\
&\quad+v_{xxxx}((k+\delta)\Delta x,s+\Delta t)\Big)+\nu\Delta t v_{xxx}(k\Delta x,s+\vartheta\Delta t)-\gamma\Delta t v_{xxxx}(k\Delta x,s+\vartheta\Delta t)\Bigg)\,\mathrm{d}s\Bigg|^2\\
&\quad+16\mathbb{E}\left|\nu\sigma\Delta t\int_{n\Delta t}^{(n+1)\Delta t}v_x(x,s)|_{x=x_k}\,\mathrm{d}W(s)\right|^2\\
&\quad+16\mathbb{E}\left|-\gamma\sigma\Delta t\int_{n\Delta t}^{(n+1)\Delta t}v_{xx}(x,s)|_{x=x_k}\,\mathrm{d}W(s)\right|^2\\
&\quad+4\sigma^2\int_{n\Delta t}^{(n+1)\Delta t}\mathbb{E}|v(x,s)|_{x=x_k}-v_k^n|^2\,\mathrm{d}s+4\sigma^2\underbrace{\int_{n\Delta t}^{(n+1)\Delta t}\mathbb{E}|v_k^n-u_k^n|^2\,\mathrm{d}s}_{\mathbb{E}|z_k^n|^2\Delta t}.
\end{aligned}$$

Therefore,

$$\mathbb{E}\left|(1+\nu\lambda+2\gamma\rho)z_k^{n+1}-(\nu\lambda+\gamma\rho)z_{k-1}^{n+1}-\gamma\rho z_{k+1}^{n+1}\right|^2\le 4(1+\sigma^2\Delta t)\sup_k\mathbb{E}|z_k^n|^2$$

$$
\begin{aligned}
&+ 8\sup_k \mathbb{E}\Bigg| -\nu \int_{n\Delta t}^{(n+1)\Delta t} \Big(\nu\Delta t v_{xx}(k\Delta x, s+\alpha\Delta t) - \gamma\Delta t v_{xxx}(k\Delta x, s+\alpha\Delta t) \\
&+ \frac{\Delta x}{2} v_{xx}((k+\eta)\Delta x, s+\Delta t)\Big)\,\mathrm{d}s \\
&+ \gamma \int_{n\Delta t}^{(n+1)\Delta t} \Big\{ -\frac{\Delta x^2}{4!}\Big(v_{xxxx}((k+\beta)\Delta x, s+\Delta t) \\
&+ v_{xxxx}((k+\delta)\Delta x, s+\Delta t)\Big) \\
&+\nu\Delta t v_{xxx}(k\Delta x, s+\vartheta\Delta t) - \gamma\Delta t v_{xxxx}(k\Delta x, s+\vartheta\Delta t)\Big\}\,\mathrm{d}s\Bigg|^2 \\
&+ 16(\nu\sigma\Delta t)^2 \sup_k \int_{n\Delta t}^{(n+1)\Delta t} \mathbb{E}|v_x(x,s)|_{x=x_k}|^2\,\mathrm{d}s \\
&+ 16(\gamma\sigma\Delta t)^2 \sup_k \int_{n\Delta t}^{(n+1)\Delta t} \mathbb{E}|v_{xx}(x,s)|_{x=x_k}|^2\,\mathrm{d}s \\
&+ 4\sigma^2 \sup_k \int_{n\Delta t}^{(n+1)\Delta t} \mathbb{E}|v(x,s)|_{x=x_k} - v_k^n|^2\,\mathrm{d}s.
\end{aligned}
$$

Let us define the following notations:

$$
\begin{cases}
\varphi_{1k} = v_{xx}(k\Delta x, s+\alpha\Delta t), \\
\varphi_{2k} = v_{xxx}(k\Delta x, s+\alpha\Delta t), \\
\varphi_{3k} = v_{xx}((k+\eta)\Delta x, s+\Delta t), \\
\varphi_{4k} = v_{xxxx}((k+\beta)\Delta x, s+\Delta t), \\
\varphi_{5k} = v_{xxxx}((k+\delta)\Delta x, s+\Delta t), \\
\varphi_{6k} = v_{xxx}(k\Delta x, s+\vartheta\Delta t), \\
\varphi_{7k} = v_{xxxx}(k\Delta x, s+\vartheta\Delta t), \\
\varphi_{8k} = v_x(x, s_k), \\
\varphi_{9k} = v_{xx}(x, s_k),
\end{cases}
$$

where the above values are finite. Based on the following inequality:

$$
\begin{aligned}
\int_{n\Delta t}^{(n+1)\Delta t} \mathbb{E}|v(x,s)|_{x=x_k} - v_k^n|^2\,\mathrm{d}s &= \mathbb{E}\int_{n\Delta t}^{(n+1)\Delta t} |v(x,s)|_{x=x_k} - v_k^n|^2\,\mathrm{d}s \\
&\le \sup_{s\in[n\Delta t,(n+1)\Delta t]} |v(x,s)|_{x=x_k} - v(k\Delta x, n\Delta t)|^2\Delta t \le \varphi_k^*\Delta t,
\end{aligned}
$$

where

$$
\varphi_k^* = \sup_{s\in[n\Delta t,(n+1)\Delta t]} |v(x,s)|_{x=x_k} - v(k\Delta x, n\Delta t)|^2.
$$

Therefore, for all k, we derive:

$$
\begin{aligned}
&\mathbb{E}\left|(1+\nu\lambda+2\gamma\rho)z_k^{n+1} - (\nu\lambda+\gamma\rho)z_{k-1}^{n+1} - \gamma\rho z_{k+1}^{n+1}\right|^2 \\
&\le 4(1+\sigma^2\Delta t)\sup_k \mathbb{E}|z_k^n|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 8\sup_k \mathbb{E}\left|\int_{n\Delta t}^{(n+1)\Delta t} \left[\left(-\nu^2 \Delta t\varphi_{1k} + \nu\gamma\Delta t\varphi_{2k} - \nu\frac{\Delta x}{2}\varphi_{3k}\right)\right.\right.\\
&+ \left.\left.\left(-\gamma\frac{\Delta x^2}{4!}(\varphi_{4k}+\varphi_{5k}) + \gamma\nu\Delta t\varphi_{6k} - \gamma^2\Delta t\varphi_{7k}\right)\right]\,\mathrm{d}s\right|^2\\
&+ 16\sup_k \int_{n\Delta t}^{(n+1)\Delta t} \left\{(\nu\sigma\Delta t)^2\mathbb{E}|\varphi_{8k}|^2 + (\gamma\sigma\Delta t)^2\mathbb{E}|\varphi_{9k}|^2\right\}\,\mathrm{d}s + 4\sigma^2\varphi_k^*\Delta t,
\end{aligned}
$$

thus,

$$
\begin{aligned}
&\sup_k \mathbb{E}\left|(1+\nu\lambda+2\gamma\rho)z_k^{n+1} - (\nu\lambda+\gamma\rho)z_{k-1}^{n+1} - \gamma\rho z_{k+1}^{n+1}\right|^2\\
&\leq 4(1+\sigma^2\Delta t)\sup_k \mathbb{E}|z_k^n|^2\\
&\quad + 8\sup_k \mathbb{E}\left|\int_{n\Delta t}^{(n+1)\Delta t} \left[\left(-\nu^2 \Delta t\varphi_{1k} + \nu\gamma\Delta t\varphi_{2k} - \nu\frac{\Delta x}{2}\varphi_{3k}\right)\right.\right.\\
&\quad + \left.\left.\left(-\gamma\frac{\Delta x^2}{4!}(\varphi_{4k}+\varphi_{5k}) + \gamma\nu\Delta t\varphi_{6k} - \gamma^2\Delta t\varphi_{7k}\right)\right]\,\mathrm{d}s\right|^2\\
&\quad + 16\sup_k \int_{n\Delta t}^{(n+1)\Delta t} \left\{(\nu\sigma\Delta t)^2\mathbb{E}|\varphi_{8k}|^2 + (\gamma\sigma\Delta t)^2\mathbb{E}|\varphi_{9k}|^2\right\}\,\mathrm{d}s + 4\sigma^2\varphi_k^*\Delta t.
\end{aligned}
$$

On the other hand, we have:

$$
\begin{aligned}
&\sup_k \mathbb{E}\left|(1+\nu\lambda+2\gamma\rho)z_k^{n+1} - (\nu\lambda+\gamma\rho)z_{k-1}^{n+1} - \gamma\rho z_{k+1}^{n+1}\right|^2\\
&\quad \geq \Big(|1+\nu\lambda+2\gamma\rho| - |\nu\lambda+\gamma\rho| - |\gamma\rho|\Big)^2 \sup_k \mathbb{E}|z_k^{n+1}|^2\\
&\quad = \sup_k \mathbb{E}|z_k^{n+1}|^2.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sup_k \mathbb{E}|z_k^{n+1}|^2 \leq{}& 4(1+\sigma^2\Delta t)\sup_k \mathbb{E}|z_k^n|^2\\
&+ 8\sup_k \mathbb{E}\left|\int_{n\Delta t}^{(n+1)\Delta t} \left[\left(-\nu^2 \Delta t\varphi_{1k} + \nu\gamma\Delta t\varphi_{2k} - \nu\frac{\Delta x}{2}\varphi_{3k}\right)\right.\right.\\
&+ \left.\left.\left(-\gamma\frac{\Delta x^2}{4!}(\varphi_{4k}+\varphi_{5k}) + \gamma\nu\Delta t\varphi_{6k} - \gamma^2\Delta t\varphi_{7k}\right)\right]\,\mathrm{d}s\right|^2\\
&+ 16(\sigma\Delta t)^2\sup_k \int_{n\Delta t}^{(n+1)\Delta t} \left\{\nu^2\mathbb{E}|\varphi_{8k}|^2 + \gamma^2\mathbb{E}|\varphi_{9k}|^2\right\}\,\mathrm{d}s\\
&+ 4\sigma^2\varphi_k^*\Delta t. \qquad (13)
\end{aligned}
$$

Setting

$$
\Omega_1 = -\nu^2\Delta t\varphi_{1k} + \nu\gamma\Delta t\varphi_{2k} - \nu\frac{\Delta x}{2}\varphi_{3k} - \gamma\frac{\Delta x^2}{4!}(\varphi_{4k}+\varphi_{5k}) + \gamma\nu\Delta t\varphi_{6k} - \gamma^2\Delta t\varphi_{7k},
$$

$\Omega_2 = 16(\sigma \Delta t)^2,$
$\Omega_3 = 4\sigma^2 \varphi_k^*.$

From the last inequality (13), it can be concluded that:

$$\sup_k \mathbb{E}|z_k^{n+1}|^2 \leq 4(1+\sigma^2 \Delta t) \sup_k \mathbb{E}|z_k^n|^2 + 8 \sup_k \mathbb{E}|\Omega_1|^2 \Delta t + \Omega_2 \sup \left(\nu^2 \mathbb{E}|\varphi_{8k}|^2 + \gamma^2 \mathbb{E}|\varphi_{9k}|^2\right) \Delta t + \Omega_3 \Delta t.$$

This results in:

$$\sup_k \mathbb{E}|z_k^{n+1}|^2 \leq 4(1+\sigma^2 \Delta t) \sup_k \mathbb{E}|z_k^n|^2 + \Phi \Delta t,$$

where

$$\Phi = 8 \sup_k \mathbb{E}|\Omega_1|^2 + \Omega_2 \sup \left(\nu^2 \mathbb{E}|\varphi_{8k}|^2 + \gamma^2 \mathbb{E}|\varphi_{9k}|^2\right) + \Omega_3. \tag{14}$$

As the step size Δt approaches zero, we see that:

$$\begin{aligned}\mathbb{E}\|z^{n+1}\|_\infty^2 &\leq 4(1+\sigma^2 \Delta t)\mathbb{E}\|z^n\|_\infty^2 + \Phi \Delta t \\ &\leq \left(1+\sigma^2 \frac{t}{n+1}\right)^{n+1} \sum_{j=1}^{n} (4\Phi \Delta t)^j + \Phi \Delta t \\ &\leq e^{\sigma^2 t} \sum_{j=1}^{n} (4\Phi \Delta t)^j + \Phi \Delta t, \end{aligned} \tag{15}$$

and consequently $\mathbb{E}\|z^{n+1}\|_\infty^2 \to 0$.
Let us examine the convergence order of the stochastic upwind scheme (3) with respect to space and time. Based on (14), we note that

$$\begin{aligned}\Phi =& 8 \sup_k \mathbb{E}|(-\nu^2 \varphi_{1k} + \nu\gamma\varphi_{2k} + \nu\gamma\varphi_{6k} - \gamma^2 \varphi_{7k})\Delta t \\ &- \frac{\nu}{2} \Delta x \varphi_{3k} - \frac{\gamma}{4!}(\varphi_{4k} + \varphi_{5k})\Delta x^2|^2 \\ &+ 16 \sup_k (\nu^2 \mathbb{E}|\varphi_{8k}|^2 + \gamma^2 \mathbb{E}|\varphi_{9k}|^2)\sigma^2 \Delta t^2 + 4\sigma^2 \varphi_k^*. \end{aligned} \tag{16}$$

By substituting (16) into (15) and noting that $t \in [0,1]$, one obtains

$$\begin{aligned}&\mathbb{E}\|z^{n+1}\|^2 \leq [e^{\sigma^2}(4\Phi)^2 + (4e^{\sigma^2}+1)\Phi]\Delta t \\ &\leq \Bigg[16e^{\sigma^2}\Bigg\{16\Delta t^2 \sup_k \mathbb{E}|(-\nu^2\varphi_{1k} + \nu\gamma\varphi_{2k} + \nu\gamma\varphi_{6k} - \gamma^2\varphi_{7k})|^2 + 2\nu^2 \Delta x^2 \sup_k \mathbb{E}|\varphi_{3k}|^2 \\ &\quad + \frac{\gamma^2}{72}\Delta x^4 \sup_k \mathbb{E}|\varphi_{4k} + \varphi_{5k}|^2 + 16\sigma^2 \Delta t^2 \sup_k (\nu^2 \mathbb{E}|\varphi_{8k}|^2 + \gamma^2 \mathbb{E}|\varphi_{9k}|^2) + 4\sigma^2 \varphi_k^*\Bigg\}^2 \\ &+ (4e^{\sigma^2}+1)\Bigg\{16\Delta t^2 \sup_k \mathbb{E}|(-\nu^2\varphi_{1k} + \nu\gamma\varphi_{2k} + \nu\gamma\varphi_{6k} - \gamma^2\varphi_{7k})|^2 + 2\nu^2 \Delta x^2 \sup_k \mathbb{E}|\varphi_{3k}|^2 \\ &\quad + \frac{\gamma^2}{72}\Delta x^4 \sup_k \mathbb{E}|\varphi_{4k} + \varphi_{5k}|^2 + 16\sigma^2 \Delta t^2 \sup_k (\nu^2 \mathbb{E}|\varphi_{8k}|^2 + \gamma^2 \mathbb{E}|\varphi_{9k}|^2) + 4\sigma^2 \varphi_k^*\Bigg\}\Bigg]\Delta t \end{aligned}$$

$$
\begin{aligned}
&= \Bigg[16e^{\sigma^2}\Bigg\{16\Delta t^2\Big[\sup_k \mathbb{E}|(-\nu^2\varphi_{1k}+\nu\gamma\varphi_{2k}+\nu\gamma\varphi_{6k}-\gamma^2\varphi_{7k})|^2+\sigma^2\sup_k(\nu^2\mathbb{E}|\varphi_{8k}|^2+\gamma^2\mathbb{E}|\varphi_{9k}|^2)\Big]\\
&\qquad + 2\nu^2\Delta x^2\sup_k\mathbb{E}|\varphi_{3k}|^2+\frac{\gamma^2}{72}\Delta x^4\sup_k\mathbb{E}|\varphi_{4k}+\varphi_{5k}|^2+4\sigma^2\varphi_k^*\Bigg\}^2\\
&+(4e^{\sigma^2}+1)\Bigg\{16\Delta t^2\Big[\sup_k \mathbb{E}|(-\nu^2\varphi_{1k}+\nu\gamma\varphi_{2k}+\nu\gamma\varphi_{6k}-\gamma^2\varphi_{7k})|^2+\sigma^2\sup_k(\nu^2\mathbb{E}|\varphi_{8k}|^2+\gamma^2\mathbb{E}|\varphi_{9k}|^2)\Big]\\
&\qquad + 2\nu^2\Delta x^2\sup_k\mathbb{E}|\varphi_{3k}|^2+\frac{\gamma^2}{72}\Delta x^4\sup_k\mathbb{E}|\varphi_{4k}+\varphi_{5k}|^2+4\sigma^2\varphi_k^*\Bigg\}\Bigg]\Delta t\\
&\le 64e^{\sigma^2}\Bigg\{16\Delta t^2\Big[\sup_k \mathbb{E}|(-\nu^2\varphi_{1k}+\nu\gamma\varphi_{2k}+\nu\gamma\varphi_{6k}-\gamma^2\varphi_{7k})|^2+\sigma^2\sup_k(\nu^2\mathbb{E}|\varphi_{8k}|^2+\gamma^2\mathbb{E}|\varphi_{9k}|^2)\Big]\Bigg\}^2\\
&\qquad + (2\nu^2\Delta x^2\sup_k\mathbb{E}|\varphi_{3k}|^2)^2+(\frac{\gamma^2}{72}\Delta x^4\sup_k\mathbb{E}|\varphi_{4k}+\varphi_{5k}|^2)^2+(4\sigma^2\varphi_k^*)^2\\
&+(4e^{\sigma^2}+1)\Bigg\{16\Delta t^2\Big[\sup_k \mathbb{E}|(-\nu^2\varphi_{1k}+\nu\gamma\varphi_{2k}+\nu\gamma\varphi_{6k}-\gamma^2\varphi_{7k})|^2+\sigma^2\sup_k(\nu^2\mathbb{E}|\varphi_{8k}|^2+\gamma^2\mathbb{E}|\varphi_{9k}|^2)\Big]\\
&\qquad + 2\nu^2\Delta x^2\sup_k\mathbb{E}|\varphi_{3k}|^2+\frac{\gamma^2}{72}\Delta x^4\sup_k\mathbb{E}|\varphi_{4k}+\varphi_{5k}|^2+4\sigma^2\varphi_k^*\Bigg\}\Bigg]\Delta t.
\end{aligned}
$$

Putting

$$
\begin{aligned}
K_1 &= 1024e^{\sigma^2}\Big[\sup_k\mathbb{E}|(-\nu^2\varphi_{1k}+\nu\gamma\varphi_{2k}+\nu\gamma\varphi_{6k}-\gamma^2\varphi_{7k})|^2\\
&\qquad +\sigma^2\sup_k(\nu^2\mathbb{E}|\varphi_{8k}|^2+\gamma^2\mathbb{E}|\varphi_{9k}|^2)\Big]^2,\\
K_2 &= 256\nu^4e^{\sigma^2}(\sup_k\mathbb{E}|\varphi_{3k}|^2)^2+\frac{\gamma^2}{72}(4e^{\sigma^2}+1)\sup_k\mathbb{E}|\varphi_{4k}+\varphi_{5k}|^2,\\
K_3 &= \frac{\gamma^4}{81}e^{\sigma^2}\left(\sup_k\mathbb{E}|\varphi_{4k}+\varphi_{5k}|^2\right)^2,\\
K_4 &= 16(4e^{\sigma^2}+1)\Big[\sup_k\mathbb{E}|(-\nu^2\varphi_{1k}+\nu\gamma\varphi_{2k}+\nu\gamma\varphi_{6k}-\gamma^2\varphi_{7k})|^2\\
&\qquad +\sigma^2\sup_k(\nu^2\mathbb{E}|\varphi_{8k}|^2+\gamma^2\mathbb{E}|\varphi_{9k}|^2)\Big],\\
K_5 &= 2\nu^2(4e^{\sigma^2}+1)\sup_k\mathbb{E}|\varphi_{3k}|^2,\\
K_6 &= 1024e^{\sigma^2}(\sigma^2\varphi_k^*)^2+4\sigma^2\varphi_k^*(4e^{\sigma^2}+1),
\end{aligned}
$$

we find that

$$
\mathbb{E}\|z^{n+1}\|^2 \le \big(K_1\Delta t^4+K_2\Delta x^4+K_3\Delta x^8+K_4\Delta t^2+K_5\Delta x^2+K_6\big)\,\Delta t.
$$

Consequently,

$$
\mathbb{E}\|z^{n+1}\|^2 = \mathrm{O}(\Delta t)+\mathrm{O}(\Delta t\Delta x^2).
$$

4 Numerical simulations

In this section, we present the numerical results of the stochastic upwind scheme (3) applied to three test problems. All numerical results were obtained using MATLAB 2018 on an Intel(R) Core(TM) i7 CPU with 12 GB RAM and a 64–bit system (Windows 10).

Example 1 Consider the following SPDE:

$$U_t(x,t) = \gamma U_{xx}(x,t) + \sigma U(x,t)\dot{W}(t), \qquad x \in [0,1], \qquad t \in [0,1], \tag{17}$$

with the following initial and boundary conditions:

$$\begin{aligned} U(x,0) &= \exp\left(-\frac{(x-0.2)^2}{\gamma}\right), && x \in [0,1],\\ U(0,t) &= \frac{1}{\sqrt{4t+1}}\exp\left(-\frac{0.04}{\gamma(4t+1)}\right), && t \in [0,1],\\ U(1,t) &= \frac{1}{\sqrt{4t+1}}\exp\left(-\frac{0.64}{\gamma(4t+1)}\right), && t \in [0,1]. \end{aligned}$$

It can be seen that, in the absence of the stochastic term, the exact solution of the problem (17) can be expressed as follows:

$$U(x,t) = \frac{1}{\sqrt{4t+1}}\exp\left(-\frac{(x-0.2)^2}{\gamma(4t+1)}\right).$$

In Figs. 1 and 2, the numerical simulations of the stochastic upwind scheme (3) are presented alongside the analytical solution for the parameter values

$$\gamma = 0.005,\ \sigma = 1,\ \Delta x = 0.01,\ \Delta t = 0.005,$$

and

$$\gamma = 0.01,\ \sigma = 0.9,\ \Delta x = 0.025,\ \Delta t = 0.01.$$

From the numerical results in Figs. 1 and 2, one can see the high accuracy of the presented scheme for solving the SPDE (17). The analytical solution and numerical solutions of the SPDE (17) using the stochastic upwind scheme (3) are shown in Figs. 3–6 on a 500×500 grid over the time interval $[0,1]$ for $\gamma = 0.005,\ \sigma = 0.1$, and $\gamma = 0.001,\ \sigma = 0.01$, respectively. Based on the numerical results obtained using the stochastic upwind scheme (3) and their comparing with the exact solution of the problem (17), it can be concluded that the stochastic upwind scheme (3) exhibits high accuracy. In Tables 1–3, the absolute errors of the stochastic upwind scheme (3) with $\gamma = 0.001,\ \sigma = 1.5,\ \Delta x = 0.01$, and $\Delta t = 0.01,\ 0.04,\ 0.05$ are compared with the stochastic difference scheme in [9]. It can be observed that the absolute errors of the stochastic upwind scheme (3) are less than those of the stochastic scheme proposed in [9].

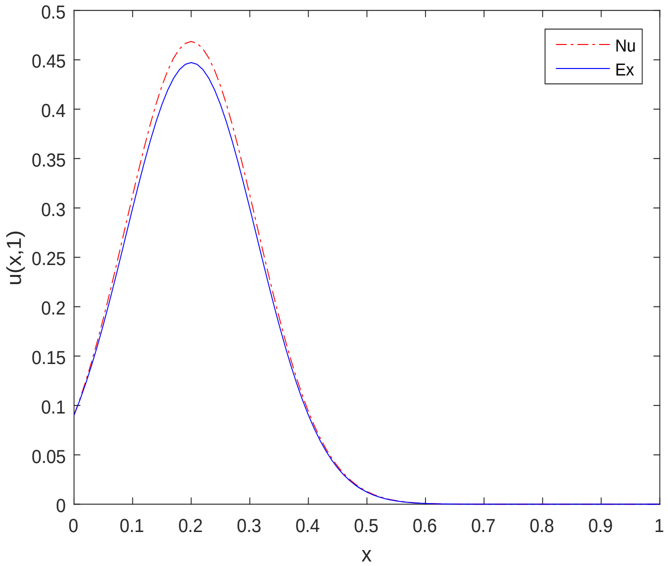

Fig. 1 Comparison between the exact solution and the stochastic upwind scheme (3) with $\gamma = 0.005$, $\sigma = 1$, $\Delta x = 0.01$ and $\Delta t = 0.005$.

Table 1 The absolute errors of the stochastic upwind scheme (3) and the proposed scheme in [9] are evaluated with parameters $\gamma = 0.001$, $\sigma = 1.5$, $\Delta x = 0.01$, and $\Delta t = 0.01$.

x	Upwind scheme	Scheme of [9]
0.1	2.1×10^{-5}	4.3000×10^{-3}
0.2	1.53×10^{-5}	3.6800×10^{-2}
0.3	2.1×10^{-5}	4.3000×10^{-3}
0.4	2.5342×10^{-7}	3.4287×10^{-5}
0.5	7.4031×10^{-11}	9.2327×10^{-9}
0.6	4.8454×10^{-17}	6.7307×10^{-14}
0.7	1.7376×10^{-23}	3.3412×10^{-20}
0.8	2.4592×10^{-32}	7.7082×10^{-28}
0.9	3.0018×10^{-40}	3.4080×10^{-36}
1	3.4864×10^{-75}	6.4864×10^{-70}

In Table 4, the CPU execution times for the stochastic upwind scheme (3) with the values $\gamma = 0.005$, $\sigma = 1$, $\Delta x = 0.01$, and $\Delta t = 0.005$ are compared to the difference schemes referenced in [8] and [9]. It can be seen that the stochastic upwind scheme (3) requires less CPU time than the cited stochastic difference schemes in [8] and [9].

Example 2 Consider the following SPDE as a second example

$$U_t(x,t) = \gamma U_{xx}(x,t) + \sigma U(x,t)\dot{W}(t), \qquad (x,t) \in [0,1] \times [0,1], \tag{18}$$

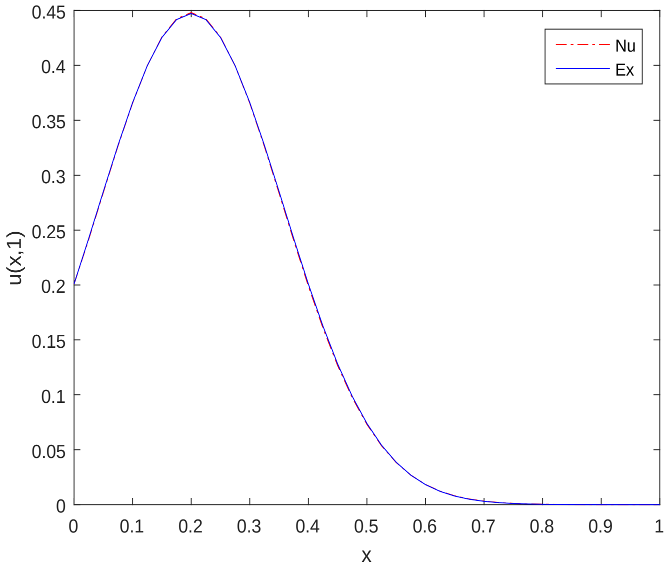

Fig. 2 Comparison between the exact solution and the stochastic upwind scheme (3) with $\gamma = 0.01$, $\sigma = 0.9$, $\Delta x = 0.025$ and $\Delta t = 0.01$.

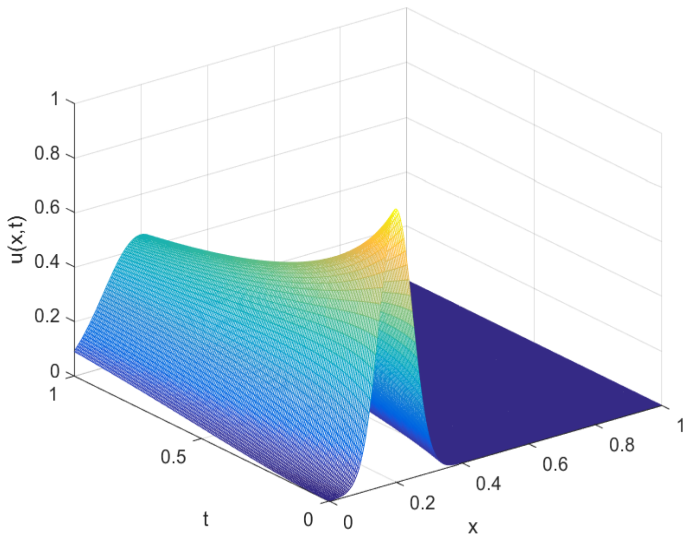

Fig. 3 The analytical solution of SPDE (17) for $\gamma = 0.005$, and $\sigma = 0.1$.

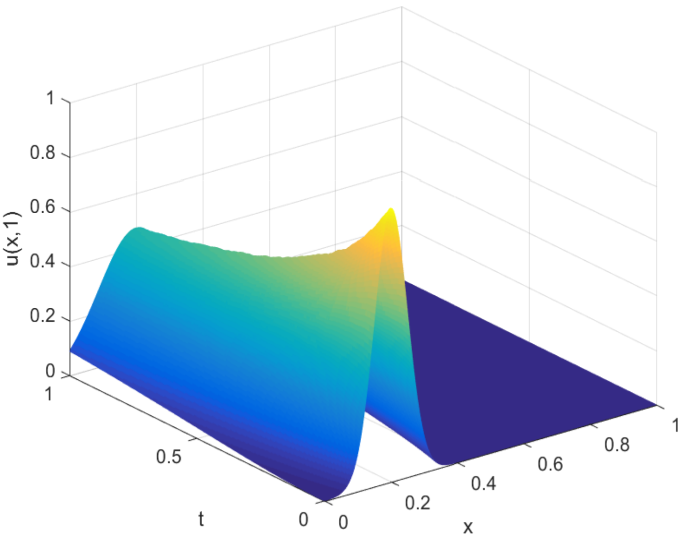

Fig. 4 The numerical solutions of the stochastic upwind scheme (3) for $\gamma = 0.005$, and $\sigma = 0.1$.

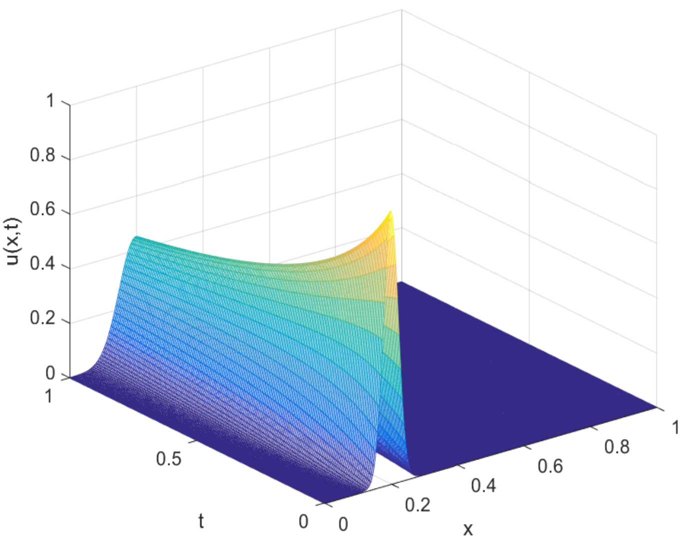

Fig. 5 The analytical solution of SPDE (17) for $\gamma = 0.001$, and $\sigma = 0.01$.

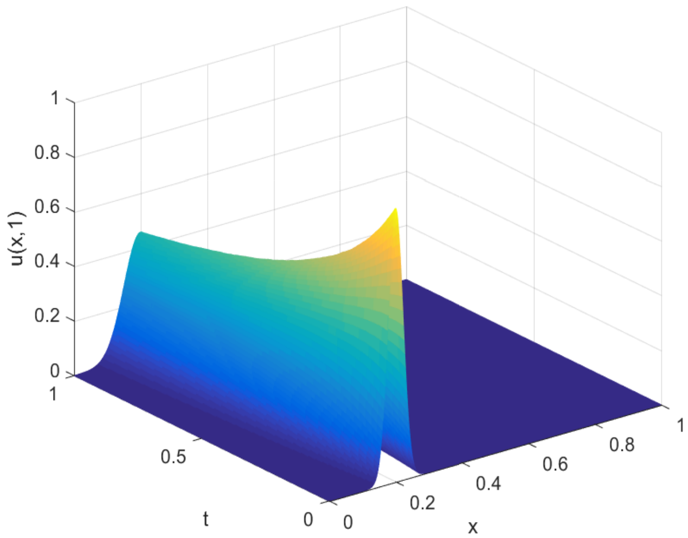

Fig. 6 The numerical solutions of the stochastic upwind scheme (3) for $\gamma = 0.001$, and $\sigma = 0.01$.

Table 2 The absolute errors of the stochastic upwind scheme (3) and the proposed scheme in [9] are evaluated with parameters $\gamma = 0.001,\ \sigma = 1.5,\ \Delta x = 0.01$, and $\Delta t = 0.04$.

x	Upwind scheme	Scheme of [9]
0.1	4.52×10^{-7}	5.8578×10^{-5}
0.2	1.55×10^{-5}	2.2000×10^{-3}
0.3	4.52×10^{-7}	5.8578×10^{-5}
0.4	4.4365×10^{-8}	5.0582×10^{-6}
0.5	6.3474×10^{-12}	7.3834×10^{-10}
0.6	6.294×10^{-18}	1.0652×10^{-15}
0.7	3.1619×10^{-26}	3.8851×10^{-22}
0.8	4.5894×10^{-34}	5.1278×10^{-30}
0.9	2.1260×10^{-41}	1.5900×10^{-37}
1	3.4864×10^{-75}	6.4864×10^{-70}

subject to the initial condition

$$U(x, 0) = \sin(\pi x), \qquad x \in [0, 1],$$

with the boundary conditions

$$U(0, t) = U(1, t) = 0, \qquad t \in [0, 1].$$

In the absence of the stochastic term, the exact solution is expressed as follows:

$$U(x, t) = e^{-\gamma \pi^2 t} \sin(\pi x).$$

Table 3 The absolute errors of the stochastic upwind scheme (3) and the proposed scheme in [9] are evaluated with parameters $\gamma = 0.001,\ \sigma = 1.5,\ \Delta x = 0.01$, and $\Delta t = 0.05$.

x	Upwind scheme	Scheme of [9]
0.1	5.2512×10^{-5}	6.0000×10^{-3}
0.2	3.66×10^{-4}	4.7400×10^{-2}
0.3	5.2512×10^{-5}	6.0000×10^{-3}
0.4	1.7317×10^{-8}	6.1739×10^{-6}
0.5	1.9297×10^{-11}	2.2026×10^{-9}
0.6	1.4632×10^{-17}	1.6669×10^{-15}
0.7	2.0693×10^{-25}	2.1245×10^{-22}
0.8	3.0994×10^{-33}	4.8151×10^{-30}
0.9	2.0264×10^{-42}	2.9264×10^{-38}
1	3.4867×10^{-75}	6.4864×10^{-70}

Table 4 Comparison of the CPU times for the stochastic upwind scheme (3) and the stochastic difference schemes in [8] and [9].

Upwind scheme	Scheme of [8]	Scheme of [9]
11.062321	18.638136	17.765299

In Figs. 7 and 8, the analytical solution and the stochastic upwind scheme (3) are compared for the values of $\gamma = 0.05,\ \sigma = 0.5,\ \Delta x = 0.005,\ \Delta t = 0.005$, and $\gamma = 0.01,\ \sigma = 1.2,\ \Delta x = \frac{1}{120},\ \Delta t = 0.01$. From the numerical results in Figs. 7 and 8, one can see the high accuracy of the stochastic upwind scheme (3) for solving the SPDE (18). Figs. 9–12 show the exact solution and the numerical solution of the SPDE (18) using the stochastic upwind scheme (3). Figs. 9 and 10, depict the results on a 500×500 grid over the time interval $[0, 1]$ for the values $\gamma = 0.05,\ \sigma = 0.01$, while Figs. 11 and 12, display the results for $\gamma = 0.01,\ \sigma = 0.06$. Based on the numerical results derived from the stochastic upwind scheme (3) and their comparison with the exact solution of the problem (18), it can be concluded that the proposed scheme demonstrates high accuracy. In Tables 5–7, the absolute errors of the stochastic upwind scheme (3) with $\gamma = 1,\ \sigma = 1,\ \Delta x = 0.01$, and $\Delta t = 0.01,\ 0.04$, and 0.05 are compared with those of the stochastic difference scheme presented in [9]. It is evident that the absolute errors associated with the stochastic upwind scheme (3) are less than those of the stochastic scheme presented in [9].

In Table 8, the CPU times of the stochastic upwind scheme (3) with parameters $\gamma = 1,\ \sigma = 1$, and $\Delta x = \Delta t = 0.01$ are compared to those of the stochastic difference schemes presented in [8] and [9]. According to the results presented in Table 8, it is evident that the CPU time for the stochastic upwind scheme is less than that of the stochastic difference schemes referenced in [8] and [9].

Example 3 Consider the following SPDE as the third example

$$U_t(x,t) + \nu U_x(x,t) = \gamma U_{xx}(x,t) + \sigma U(x,t)\dot{W}(t), \qquad (x,t) \in [0,1] \times [0,1], \quad (19)$$

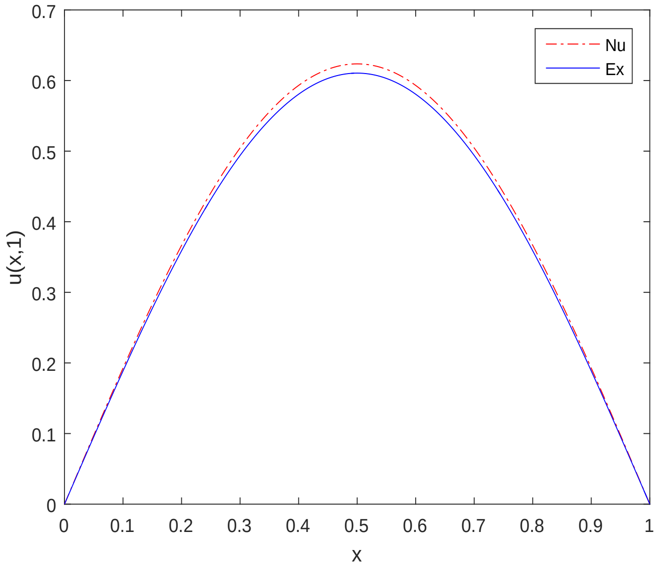

Fig. 7 Comparison between the exact solution and the stochastic upwind scheme (3) with $\gamma = 0.05$, $\sigma = 0.5$, $\Delta x = 0.005$ and $\Delta t = 0.005$.

Table 5 Absolute errors of the stochastic upwind scheme (3) and the scheme mentioned in [9] are reported for $\gamma = \sigma = 1$, and $\Delta x = \Delta t = 0.01$.

x	Upwind scheme	Scheme of [9]
0.1	1.2341×10^{-9}	2.2049×10^{-7}
0.2	8.7736×10^{-9}	4.1940×10^{-7}
0.3	4.4411×10^{-9}	5.7726×10^{-7}
0.4	5.8697×10^{-9}	6.7861×10^{-7}
0.5	6.0174×10^{-9}	7.1353×10^{-7}
0.6	5.8697×10^{-9}	6.7861×10^{-7}
0.7	4.4411×10^{-9}	5.7726×10^{-7}
0.8	8.7736×10^{-9}	9.1946×10^{-7}
0.9	1.2341×10^{-9}	2.2049×10^{-7}
1	6.3343×10^{-23}	6.3343×10^{-21}

with the following initial and boundary conditions:

$$
\begin{aligned}
U(x,0) &= \exp\left(-\frac{(x-0.5)^2}{\gamma}\right), && x \in [0,1],\\
U(0,t) &= \frac{1}{\sqrt{4t+1}} \exp\left(-\frac{(0.5-\nu t)^2}{\gamma(4t+1)}\right), && t \in [0,1],\\
U(1,t) &= \frac{1}{\sqrt{4t+1}} \exp\left(-\frac{(0.5-\nu t)^2}{\gamma(4t+1)}\right), && t \in [0,1].
\end{aligned}
$$

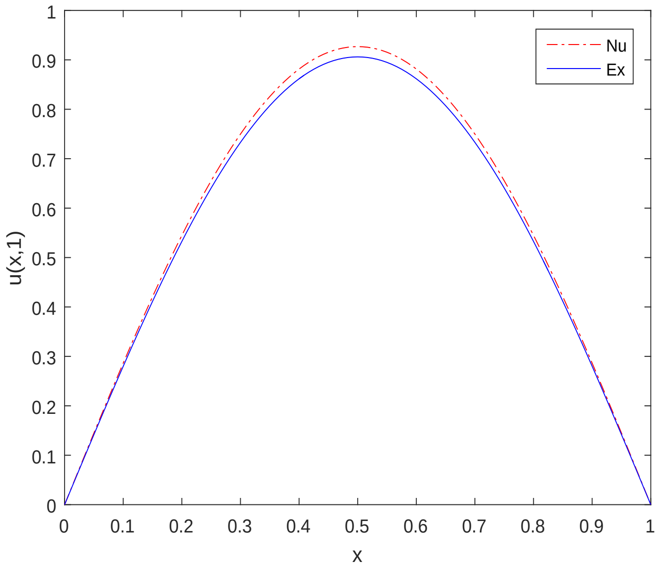

Fig. 8 Comparison between the exact solution and the stochastic upwind scheme (3) with $\gamma = 0.01$, $\sigma = 1.2$, $\Delta x = 1/120$ and $\Delta t = 0.01$.

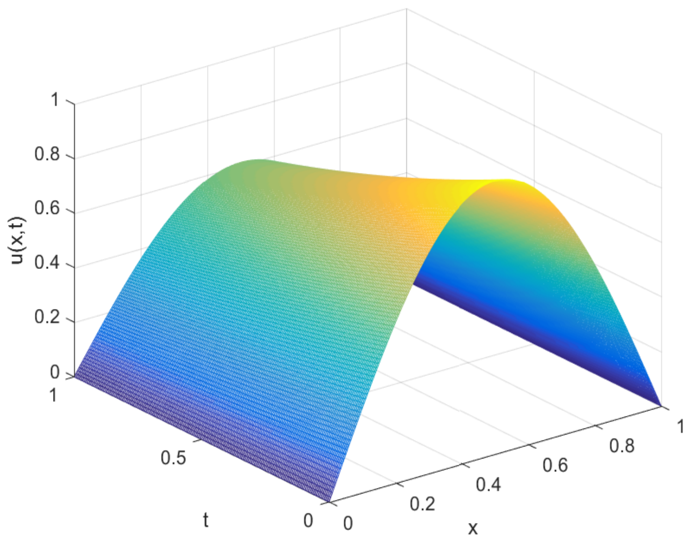

Fig. 9 The analytical solution of the SPDE (18) for $\gamma = 0.05,\ \sigma = 0.01$.

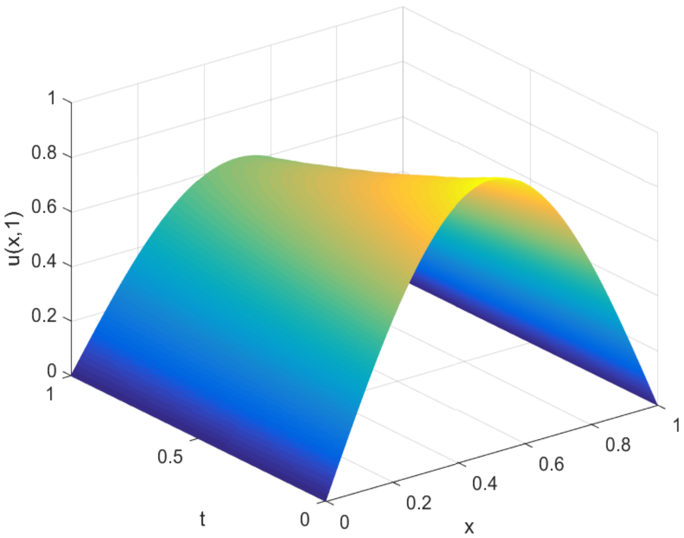

Fig. 10 The numerical solutions of the stochastic upwind scheme (3) for $\gamma = 0.05,\ \sigma = 0.01$.

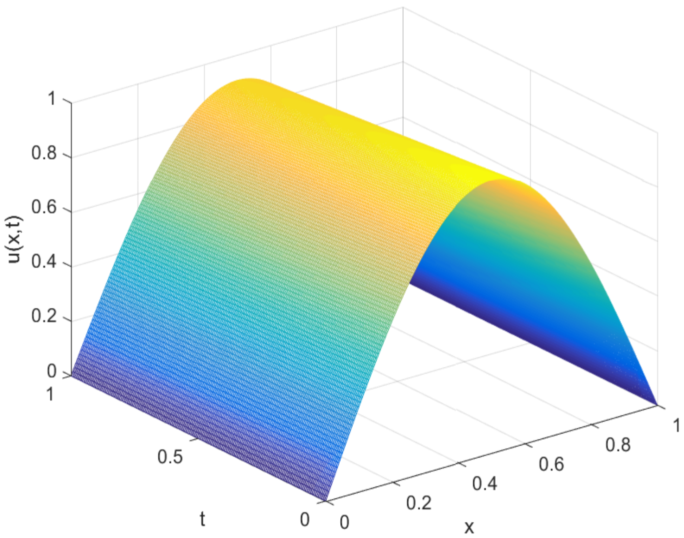

Fig. 11 The analytical solution of the SPDE (18) for $\gamma = 0.01,\ \sigma = 0.06$.

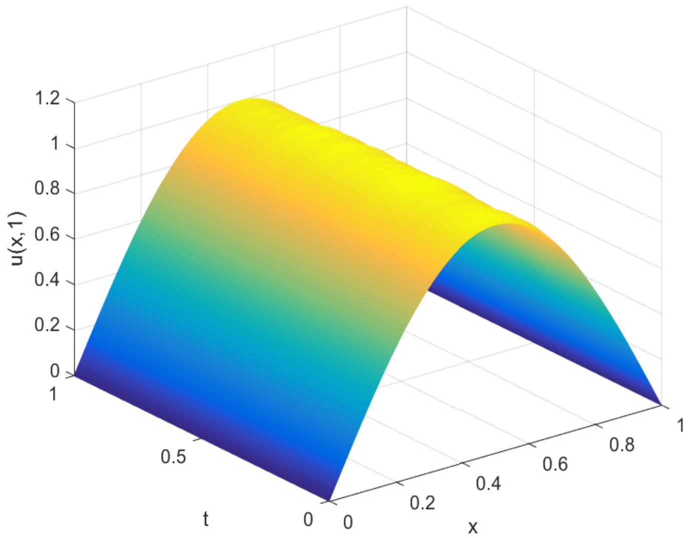

Fig. 12 The numerical solutions of the stochastic upwind scheme (3) for $\gamma = 0.01,\ \sigma = 0.06$.

Table 6 Absolute errors of the stochastic upwind scheme (3) and the scheme mentioned in [9] are presented with $\gamma = 1,\ \sigma = 1,\ \Delta x = 0.01$, and $\Delta t = 0.04$.

x	Upwind scheme	Scheme of [9]
0.1	2.1700×10^{-9}	7.2438×10^{-7}
0.2	1.276×10^{-8}	1.3779×10^{-6}
0.3	1.1711×10^{-8}	1.8965×10^{-6}
0.4	1.6785×10^{-8}	2.2294×10^{-6}
0.5	2.0222×10^{-8}	2.3441×10^{-6}
0.6	1.6785×10^{-8}	2.2294×10^{-6}
0.7	1.1711×10^{-8}	1.8965×10^{-6}
0.8	1.276×10^{-8}	1.3779×10^{-6}
0.9	2.1700×10^{-9}	7.2438×10^{-7}
1	6.3343×10^{-23}	6.3343×10^{-21}

It is straightforward to demonstrate that, without the stochastic term, the exact solution (19) can be expressed as follows:

$$U(x,t) = \frac{1}{\sqrt{4t+1}} \exp\left(-\frac{(x-0.5-\nu t)^2}{\gamma(4t+1)}\right).$$

In Figs. 13 and 14, the stochastic upwind scheme (3) is compared with the exact solution for the values $\gamma = 0.005,\ \nu = 0.1,\ \sigma = 1,\ \Delta x = 0.01,\ \Delta t = 0.005$, and $\gamma = 0.05,\ \nu = 0.05,\ \sigma = 1.2,\ \Delta x = \Delta t = 0.01$. The numerical results presented in Figs. 13 and 14 demonstrate the high accuracy of the proposed scheme for solving the SPDE (19). In Figs. 15–18, we plot the exact solution

Table 7 Absolute errors of the stochastic upwind scheme (3) and the scheme mentioned in [9] are evaluated for $\gamma = 1,\ \sigma = 1,\ \Delta x = 0.01$, and $\Delta t = 0.05$.

x	Upwind scheme	Scheme of [9]
0.1	1.001×10^{-8}	1.0274×10^{-6}
0.2	1.0025×10^{-8}	1.9542×10^{-6}
0.3	2.1235×10^{-8}	2.6897×10^{-6}
0.4	2.0041×10^{-8}	3.1619×10^{-6}
0.5	2.9564×10^{-8}	3.3247×10^{-6}
0.6	2.0041×10^{-8}	3.1619×10^{-6}
0.7	2.1235×10^{-8}	2.6897×10^{-6}
0.8	1.0025×10^{-8}	1.9542×10^{-6}
0.9	1.001×10^{-8}	1.0274×10^{-6}
1	6.3343×10^{-23}	6.3343×10^{-21}

Table 8 Comparison of CPU times for the stochastic upwind scheme (3) with the stochastic difference schemes mentioned in [8,9].

Upwind scheme (3)	Scheme of [8]	Scheme of [9]
7.505272	13.865644	12.189545

alongside the approximate solution for the SPDE (19), obtained using the stochastic upwind scheme on a 500×500 grid with parameters $\gamma = 0.005,\ \nu = 0.1,\ \sigma = 0.03$, and $\gamma = \nu = 0.05,\ \sigma = 0.07$ over the time interval $[0, 1]$. Based on the numerical solutions obtained from the stochastic upwind scheme (3) and their comparison with the exact solution to problem (19), it is evident that the proposed stochastic scheme is effective and reliable. The absolute errors of the stochastic upwind scheme (3) with parameters $\gamma = 0.01,\ \nu = 0.03,\ \sigma = 2,\ \Delta x = 0.01$, and $\Delta t = 0.0001,\ 0.0002,\ 0.001$ are reported in Tables 9–11. It is apparent that the absolute errors associated with the stochastic upwind scheme (3) are less than those of the stochastic scheme analyzed in [10].

Table 9 The absolute errors of the stochastic upwind scheme (3) and the proposed scheme in [10] are reported with parameters $\gamma = 0.01,\ \nu = 0.03,\ \sigma = 2,\ \Delta x = 0.01$, and $\Delta t = 0.0001$.

x	Upwind scheme	Scheme of [10]
0.1	2.1219×10^{-6}	3.12×10^{-4}
0.2	3.1835×10^{-6}	2.8512×10^{-3}
0.3	5.3519×10^{-6}	5.1714×10^{-3}
0.4	6.1723×10^{-6}	9.93×10^{-4}
0.5	4.3891×10^{-6}	8.1639×10^{-4}
0.6	7.1349×10^{-6}	5.3542×10^{-4}
0.7	6.2790×10^{-6}	5.7891×10^{-4}
0.8	3.7517×10^{-6}	3.1435×10^{-4}
0.9	3.1728×10^{-6}	6.32×10^{-4}
1	1.0408×10^{-17}	1.133×10^{-14}

In Table 12, the CPU times for the stochastic upwind scheme (3) and the proposed stochastic scheme in [8] are reported for the parameters $\gamma = 0.005,\ \nu =$

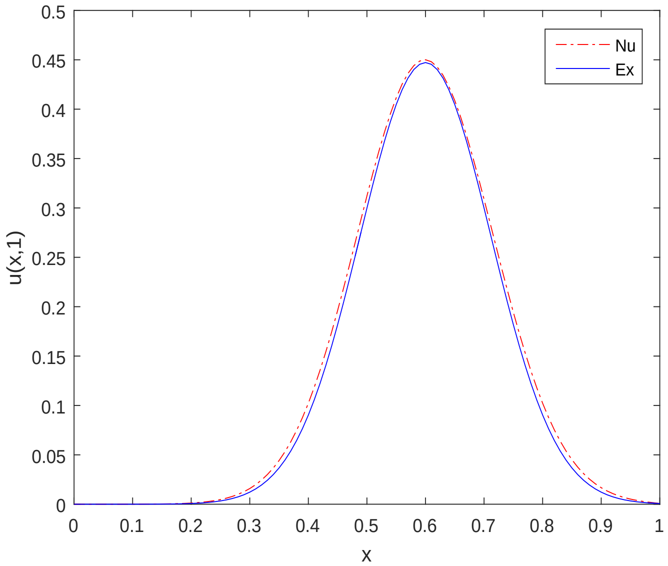

Fig. 13 Comparison of the exact solution and the stochastic numerical solution of (19) with the parameters $\gamma = 0.005$, $\nu = 0.1$, $\sigma = 1$, $\Delta x = 0.01$, and $\Delta t = 0.005$.

Table 10 The absolute errors of the stochastic upwind scheme (3) and the proposed scheme in [10] are evaluated using the parameters $\gamma = 0.01$, $\nu = 0.03$, $\sigma = 2$, $\Delta x = 0.01$, and $\Delta t = 0.0002$.

x	Upwind scheme	Scheme of [10]
0.1	3.7125×10^{-4}	5.49×10^{-2}
0.2	3.1719×10^{-4}	6.4860×10^{-2}
0.3	3.5325×10^{-4}	8×10^{-2}
0.4	2.1214×10^{-3}	3.3481×10^{-1}
0.5	2.7823×10^{-3}	3×10^{-1}
0.6	3.2514×10^{-3}	4.102×10^{-1}
0.7	2.32×10^{-3}	5.12×10^{-1}
0.8	3×10^{-4}	4.8263×10^{-2}
0.9	3.1728×10^{-7}	6.32×10^{-4}
1	1.0408×10^{-17}	1.0133×10^{-15}

0.1, $\sigma = 1$, $\Delta x = 0.01$, and $\Delta t = 0.005$. The results presented in this table indicate that the stochastic upwind scheme (3) requires less CPU time than the stochastic difference scheme referenced in [8].

5 Conclusion

In this study, we developed a stochastic upwind scheme for the numerical solutions of Itô stochastic advection–diffusion partial differential equations. We

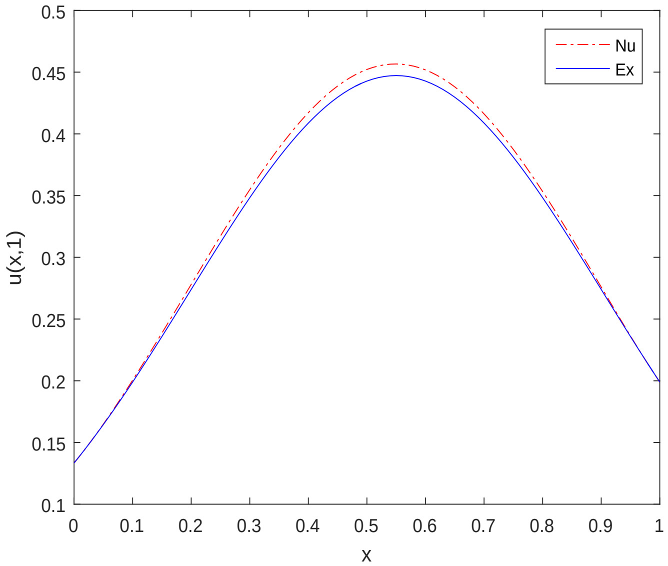

Fig. 14 Comparison of the exact solution and the stochastic numerical solution of (19) with the parameters $\gamma = \nu = 0.05$, $\sigma = 1.2$, $\varDelta x = 0.01$, and $\varDelta t = 0.01$.

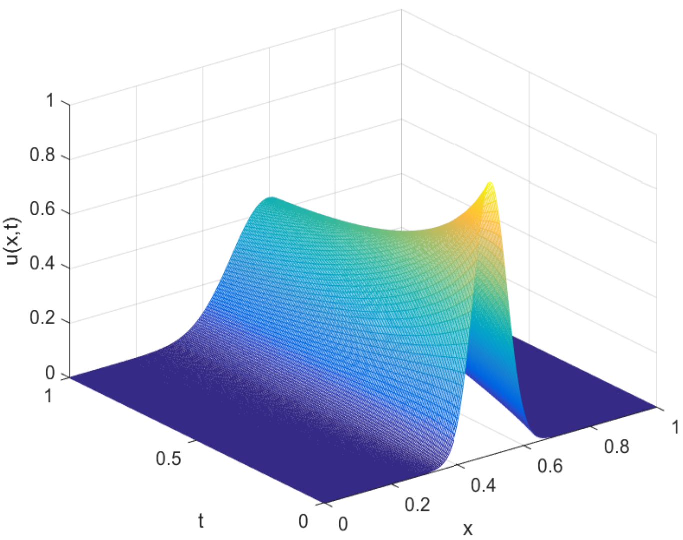

Fig. 15 The exact solution for SPDE (19) with the parameters $\gamma = 0.005$, $\nu = 0.1$, and $\sigma = 0.03$.

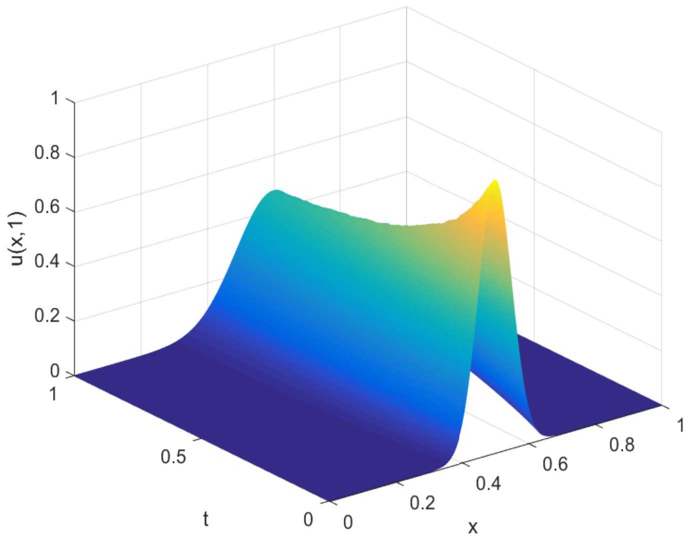

Fig. 16 The numerical solutions of the stochastic upwind scheme (3) for the parameters $\gamma = 0.005,\ \nu = 0.1$, and $\sigma = 0.03$.

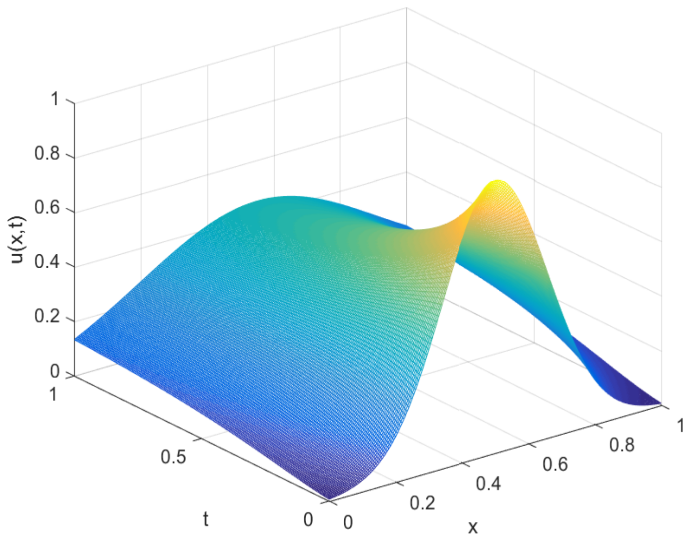

Fig. 17 The exact solution for SPDE (19) with values $\gamma = \nu = 0.05,\ \sigma = 0.07$.

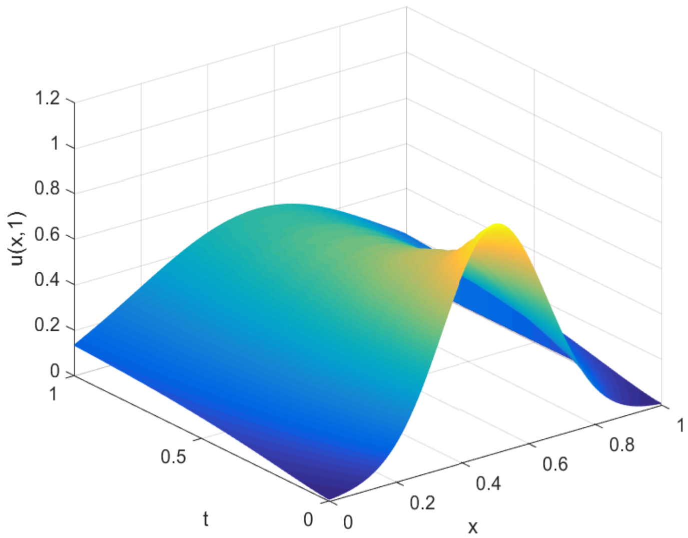

Fig. 18 The numerical solutions is obtained from the stochastic upwind scheme (3) with the parameters $\gamma = \nu = 0.05$, and $\sigma = 0.07$.

Table 11 The absolute errors of the stochastic upwind scheme (3) and the proposed scheme in [10] are evaluated with $\gamma = 0.01,\ \nu = 0.03,\ \sigma = 2,\ \Delta x = 0.01$, and $\Delta t = 0.001$.

x	Upwind scheme	Scheme of [10]
0.1	8.8098×10^{-4}	3.2204×10^{-2}
0.2	3.1×10^{-3}	7.0249×10^{-1}
0.3	7.4×10^{-3}	9.2512×10^{-1}
0.4	1.27×10^{-3}	1.8256×10^{-1}
0.5	1.6×10^{-3}	2.2549×10^{-1}
0.6	1.5×10^{-3}	3.49×10^{-1}
0.7	1.05×10^{-3}	8.2549×10^{-1}
0.8	5.4×10^{-3}	5.48×10^{-1}
0.9	2×10^{-3}	5.24×10^{-1}
1	1.0408×10^{-17}	1.0133×10^{-15}

Table 12 Comparison of CPU times between the stochastic upwind scheme (3) and the stochastic difference scheme presented in [8].

Upwind scheme	Scheme of [8]
7.497430	11.190063

then investigated the consistency, unconditional stability, and convergence of the proposed stochastic scheme. Afterward, we determined the order of convergence of the scheme with respect to space and time. Finally, to ascertain the accuracy and effectiveness of the introduced stochastic scheme, we presented three test problems with different initial and boundary conditions and compared the absolute errors and CPU times of our proposed scheme with those of existing stochastic schemes. The numerical simulations demonstrated that implementing the stochastic upwind scheme, in comparison to several existing stochastic difference schemes, resulted in reduced CPU time and fewer absolute errors when applied to stochastic advection–diffusion partial differential equations. Future work can be focused on developing effective nonstandard finite difference schemes for SPDEs and exploring robust numerical methods for solving Itô stochastic fractional–order partial differential equations.

References

1. M. Zakai, On the optimal filtering of diffusion processes, Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete, 11(3), 230–43 (1969).
2. R. Mikulevicius, BL. Rozovskii, Stochastic Navier–Stokes equations for turbulent flows, SIAM Journal on Mathematical Analysis, 35(5), 1250–310 (2004).
3. DA. Dawson, EA. Perkins, Measure-valued processes and renormalization of branching particle systems, Mathematical Surveys and Monographs, 64, 45–106 (1998).
4. L. Roques, D. Allard, S. Soubeyrand, Spatial statistics and stochastic partial differential equations: A mechanistic viewpoint. Spatial Statistics, 50, 100591 (2022).
5. JD. Murray, Mathematical biology: II: spatial models and biomedical applications, Springer, New York, 2003.
6. EJ. Allen, SJ. Novosel, Z. Zhang, Finite element and difference approximation of some linear stochastic partial differential equations, Stochastics: An International Journal of Probability and Stochastic Processes, 64(1-2), 117–42 (1998).
7. M. Namjoo, A. Mohebbian, Approximation of stochastic advection–diffusion equations with finite difference scheme, Journal of Mathematical Modeling, 4(1), 1–8 (2016).
8. M. Namjoo, A. Mohebbian, Analysis of the stability and convergence of a finite difference approximation for stochastic partial differential equations, Computational Methods for Differential Equations, 7(3), 334–58 (2019).
9. D. Baleanu, M. Namjoo, A. Mohebbian, A. Jajarmi, A weighted average finite difference scheme for the numerical solution of stochastic parabolic partial differential equations, CMES-Computer Modeling in Engineering & Sciences, 135(2), (2023).
10. M. Karami, A. Mohebbian, S. Razaghian, M. Namjoo, M. Aminian, Numerical solutions for a class of stochastic patial differential equations, Journal of Mahani Mathematical Research Center, 13(1), (2024).
11. P. E. Kloeden, E. Platen, Numerical solution of stochastic differential equations, Springer Science & Business Media, 2013.
12. M. Bishehniasar, AR. Soheili, Approximation of stochastic advection-diffusion equation using compact finite difference technique, Iranian Journal of Science, 37(3), 327–33 (2013).
13. C. Roth, Difference methods for stochastic partial differential equations, Journal of Applied Mathematics and Mechanics, 82(11-12), 821–30 (2002).
14. M. Dehghan, Weighted finite difference techniques for the one-dimensional advection–diffusion equation, Applied Mathematics and Computation, 147(2), 307–19 (2004).
15. J. Wang, X. Pang, F. Yin, J. Yao, A deep neural network method for solving partial differential equations with complex boundary in groundwater seepage, Journal of Petroleum Science and Engineering, 209, 109880 (2022).
16. G. Prato, L. Tubaro, Stochastic partial differential equations and applications, Springer Berlin Heidelberg, 1987.
17. MW. Yasin, MS. Iqbal, N. Ahmed, A. Akgül, A. Raza, M. Rafiq, MB. Riaz, Numerical scheme and stability analysis of stochastic Fitzhugh–Nagumo model, Results in Physics, 32, 105023 (2022).

18. N. Kaur, K. Goyal, An adaptive wavelet optimized finite difference B-spline polynomial chaos method for random partial differential equations, Applied Mathematics and Computation, 415, 126738 (2022).
19. L. Guo, H. Wu, T. Zhou, Normalizing field flows: Solving forward and inverse stochastic differential equations using physics-informed flow models, Journal of Computational Physics, 461, 111202 (2022).
20. NH. Sweilam, DM. ElSakout, MM. Muttardi, High-resolution schemes for stochastic nonlinear conservation laws, International Journal of Applied and Computational Mathematics, 6, 1–23 (2020).
21. NH. Sweilam, DM. El-Sakout, MM. Muttardi, Compact finite difference method to numerically solving a stochastic fractional advection-diffusion equation, Advances in Difference Equations, 2020, 1–20 (2020).
22. JW. Thomas, Numerical partial differential equations: finite difference methods, Springer Science & Business Media, 2013.
23. Khan MA, Ullah S, Kumar S, A robust study on 2019-nCOV outbreaks through non-singular derivative, The European Physical Journal Plus, 136, 1-20 (2021).
24. Kumar S, Kumar A, Samet B, Dutta H, A study on fractional host–parasitoid population dynamical model to describe insect species, Numerical Methods for Partial Differential Equations, 37(2), 1673-92 (2021).
25. Kumar S, Chauhan RP, Momani S, Hadid S, Numerical investigations on COVID-19 model through singular and non-singular fractional operators. Numerical Methods for Partial Differential Equations, 40(1), e22707 (2024).
26. Ghanbari B, Kumar S, A study on fractional predator–prey–pathogen model with Mittag–Leffler kernel-based operators, Numerical Methods for Partial Differential Equations, 40(1), e22689 (2024).
27. Kumar S, Kumar R, Momani S, Hadid S, A study on fractional COVID-19 disease model by using Hermite wavelets, Mathematical Methods in the Applied Sciences, 46(7), 7671-87 (2023).
28. Veeresha P, Prakasha DG, Kumar S, A fractional model for propagation of classical optical solitons by using nonsingular derivative, Mathematical Methods in the Applied Sciences (2020).