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An Extension of Order Bounded Operators

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Abstract Let E be a normed lattice and an g-order dense majorizing sublattice of a vector lattice E^t. We extend the norm of E to E^t, denoted by $\|.\|_t$. The pair $(E^t, \|.\|_t)$ forms a normed lattice and preserves certain lattices and topological properties whenever these properties hold in E. As a consequence, every positive linear operator defined on a normed lattice E has a linear extension to E^t. This manuscript provides an explicit formula for these extensions. The extended operator T^t is a lattice homomorphism from E^t into F, and it belongs to $\mathcal{L}_n(E^t, F)$ whenever $0 \leq T \in \mathcal{L}_n(E, F)$ and $T(x \wedge y) = Tx \wedge Ty$ for all $0 \leq x, y \in E$. Furthermore, if $T \in \mathcal{L}_b(E, F)$ and certain lattice and topological properties hold for T, then $T^t \in \mathcal{L}_b(E^t, F)$ will also preserve these properties.

Keywords Riesz space · Order convergence · Unbounded order convergence

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1 Introduction

A vector sublattice E of vector lattice G is said to be order dense in G whenever for each $0 < x \in G$ there exists some $y \in E$ with $0 < y \leq x$ and E is generalized

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order dense (g-order dense) in G whenever for each $0 < x < z$ in G there exists some $y \in E$ with $0 < x \leq y \leq z$. It is clear that each g-order dense subspace is order dense, but the converse not holds. For example, c_0 is order dense in ℓ^∞, but is not g-order dense. Let us say that a vector subspace E of an ordered vector space G is majorizing of G whenever for each $x \in G$ there exists some $y \in E$ with $x \leq y$. Let E be a normed lattice that is both g-order dense and majorizing in a vector lattice E^t. It is possible to extend the norm from E to E^t. In this paper, we investigate the method of this norm extension and demonstrate that certain lattice and topological properties can be carried over from E to E^t. Now, suppose T is a positive order bounded operator from a normed lattice E to a Dedekind complete normed lattice F. Then, there exists a linear operator T^t from E^t to F that extends T, and furthermore, we have $\|T\| = \|T^t\|$. In Section 1.2 of [2], the authors studied some new extensions of operators on vector lattices. In [3], Onno van Gaans introduced and studied a generalization of the notion of a seminorm on a directed partially ordered vector space. In this paper, we investigate this problem in a different way and extend some results to the general case.

Let E be a normed lattice and a sublattice of G, and assume that E is order dense and majorizing in a vector lattice E^t that is a subset of G. The motivations of this manuscript are as follows:

1. We can extend the norm from E to E^t as follows: For any $x \in E^t$, we define $\|x\|_t = \inf\{\|y\| : y \in E, y \geq |x|\}$, where $|x| = x \vee (-x)$, which is the supremum of x and its additive inverse $-x$. Then, $(E^t, \|\cdot\|_t)$ is a normed lattice.
2. Suppose T is an order-bounded operator from E to a Dedekind complete normed lattice F. We can define a linear extension $T^t : E^t \to F$ of T to E^t as follows:
   For any $x \in E^t$, we define $T^t(x) = \sup\{T(y) : y \in E, y \leq x\}$, where the supremum is taken in F. Then, T^t is well-defined and order-bounded.
3. Moreover, T^t is the unique linear extension of T from E^t to F in the sense that if $S : E^t \to F$ is any extension of T using the same method, then $T^t = S$.
   If certain lattice and topological properties hold for $T \in \mathcal{L}_b(E, F)$, then $T^t \in \mathcal{L}_b(E^t, F)$ will also preserve these properties.

To state our result, we need to fix some notation and recall some definitions. A Banach lattice E has order continuous norm if $\|x_\alpha\| \to 0$ for every decreasing net $(x_\alpha)_\alpha$ with $\inf_\alpha x_\alpha = 0$. A Banach lattice E is said to be an AL-space if we have $\|x + y\| = \|x\| + \|y\|$ for each $x, y \in E$ such that $|x| \wedge |y| = 0$. A Banach lattice E is said to be KB-space whenever each increasing norm bounded sequence of E^+ is norm convergent. A Riesz space that is at the same time Dedekind complete and laterally complete is referred to as a universally complete Riesz space. Let E and F be Riesz spaces. An operator $T : E \to F$ is said to be order bounded if it maps each order bounded subset of E into order bounded subset of F. The collection of all order bounded operators from a Riesz space E into a Riesz space F will be denoted by $\mathcal{L}_b(E, F)$. A linear operator between two Riesz spaces is order continuous (resp. σ-order continuous) if it maps order null nets (resp. sequences) to order null nets (resp. sequences). The collection of all order continuous (resp. σ-order continuous) linear operators from vector lattice E into vector lattice F will be denoted by $\mathcal{L}_n(E,F)$ (resp. $\mathcal{L}_c(E,F)$).

A Dedekind complete vector lattice G is said to be a Dedekind completion of the vector lattice E whenever E is lattice isomorphism to a majorizing order dense sublattice of G. A subset A of a vector lattice E is said to be order closed whenever $(x_\alpha)_\alpha \subseteq A$ and $x_\alpha \xrightarrow{o} x$ in E imply $x \in A$. A lattice norm $\|.\|$ on a vector lattice E is said to be a Fatou norm (or that $\|.\|$ satisfies the Fatou property) if $0 \leq x_\alpha \uparrow x$ in E implies $\|x_\alpha\| \uparrow \|x\|$. σ-Fatou norm has similar definition. An operator $T : E \to E$ on a vector lattice is said to be band preserving whenever T leaves all bands of E invariant, i.e., whenever $T(B) \subseteq B$ holds for each band B of E. An operator $T : E \to F$ between two vector lattices is said to be preserve disjointness whenever $x \perp y$ in E implies $Tx \perp Ty$ in F. For a normed lattice E, E' is the its order dual and $\sigma(E,E')$ is the weak topology for E. For unexplained terminology and facts on Banach lattices and positive operators, we refer the reader to [1,2].

2 An extension of the norms

Let E be an Archimedean vector lattice. Then there exists a Dedekind complete vector lattice E^δ that contains a majorizing, order dense vector subspace that is Riesz isomorphic to E, which we will identify as E. E^δ is called the Dedekind completion of E. Throughout this manuscript, we assume that the vector lattices under consideration are Archimedean. Let E and G be a normed lattice and a vector lattice, respectively, such that E is order dense and majorizing in G. The universal completion of a vector lattice E will be denoted by E^u. According to [[1], Theorem 7.21], every Archimedean vector lattice has a unique universal completion. In all parts of this manuscript, we assume that E is g-order dense and majorizing in G. Throughout this paper, $(E, \|\cdot\|)$ denotes a normed space that serves as a vector sublattice of G.

Theorem 1 *For each $x \in G$, let $\rho(x) = \sup\{\|z\| : \ z \leq |x|, \ z \in E^+\}$. Then $\rho(x)$ is a norm on G, and moreover, $(G, \rho(x))$ is a normed lattice.*

Proof It is clear that $\rho(x) = 0$ if and only if $x = 0$, and $\rho(\lambda x) = |\lambda|\rho(x)$ for each real number λ and $x \in G$. Now we prove that $\rho(x+y) \leq \rho(x) + \rho(y)$ whenever $x, y \in G$.

Let $x, y \in G$. Fix $z \in E^+$ such that $z \leq |x+y|$. By Riesz Decomposition property, [[1], Theorem 1.10], there are $z_1, z_2 \in G$ such that $|z_1| \leq |x|$, $|z_2| \leq |y|$ and $z = z_1 + z_2$. Since E is order dense in G, there are $w_1, w_2 \in E^+$ such that $|z_1| \leq w_1 \leq |x|$ and $|z_2| \leq w_2 \leq |y|$. It follows that

$$z = z_1 + z_2 \leq |z_1| + |z_2| \leq w_1 + w_2 \leq |x| + |y|.$$

Then we have

$$\|z\| \leq \|w_1 + w_2\| \leq \|w_1\| + \|w_2\| \leq \rho(x) + \rho(y).$$

Consequently, we have $\sup\{\|z\| : \ z \leq |x+y| \text{ and } z \in E^+\} \leq \rho(x) + \rho(y)$, which implies that $\rho(x+y) \leq \rho(x) + \rho(y)$.

For a normed lattice $(E, \|.\|)$, assume that E^ρ is the set of all $x \in G$ such that satisfies in the following equality,

$$\rho(x) = \inf\{\|y\| : \ |x| \leq y, \ y \in E^+\} \tag{1}$$

$$= \sup\{\|z\| : \ z \leq |x|, \ z \in E^+\}. \tag{2}$$

Then E is subspace of E^ρ and ρ is a real function from E^ρ into $[0, +\infty)$ and satisfies in the following properties:

1. $\rho(x) = 0$ iff $x = 0$
2. $\rho(\lambda x) = \lambda \rho(x)$ for each $\lambda \in \mathbb{R}^+$ and $x \in E^\rho$.
3. $\rho(x+y) \leq \rho(x) + \rho(y)$, for $x, y \in E^\rho$.

(E^ρ, ρ) is an extension of $(E, \|\cdot\|)$, meaning that E is a sublattice of E^ρ and $\|x\| = \rho(x)$ for all $x \in E$.
To see why this is true, note that by Theorem 1, we can extend the norm on E to a complete lattice norm ρ on E^ρ, such that $\|x\| = \rho(x)$ for all $x \in E$. Therefore, (E^ρ, ρ) is indeed an extension of $(E, \|\cdot\|)$.
An example that illustrates this point is as follows.

Example 1 Let c be the collection of all real number sequences which are convergence in $\mathbb{R}$ with ℓ^∞-norm. It is obvious that c is order dense majorizing of ℓ^∞. By easy calculation, we can prove that $c^\rho = \ell^\infty$.

Definition 1 Assume that $E \subseteq E^t$ is a vector sublattice of G in which every element of E^t satisfies the equalities (1) and (2), we can define a new norm in E^t called the t-norm, denoted by $\|x\|_t = \rho(x)$.

It is evident that $(E^t, \|.\|_t)$ is a normed lattice. However, E^t is not necessarily unique, and in general, we have $E \subseteq E^t \subseteq G$. The objective of this manuscript is to identify vector lattices E^t that are distinct from E. Therefore, in this manuscript, E is a proper sublattice of E^t.
In Theorem 2, we will demonstrate that $E^t = G$ whenever E is a Dedekind complete or has an order-continuous norm.

Theorem 2 *By one of the following conditions, the equality* (1) *holds for each* $x \in G$*, that is,* $E^t = G$*,* $(G, \|.\|_t)$ *is normed lattice and* $\|y\| = \|y\|_t$ *for each* $y \in E$*.*

i) E is a Dedekind complete.
ii) E has order continuous norm.

Proof i) According to Theorem 1, the function

$$\rho(x) = \sup\{\|z\| : z \leq |x|,\, z \in E^+\},$$

defines a norm for the vector lattice G. By contradiction, assume that

$$\rho(x) < \inf\{\|y\| : \ |x| \leq y,\ y \in E^+\}.$$

Let $A = \{y \in E^+ : \ |x| \leq y\}$. Since E is order dense in G, A is bounded below, and so A has infimum in E, by Dedekind completeness of E. Take $\inf A = y_0$ where $y_0 \in E$. It is clear that $y_0 < |x|$ and $\rho(x) \leq \|y_0\|$. Then $\|y_0\| = \rho(y_0) = \rho(x)$. Let the natural number n be enough large such that

$$\rho(x) < \|y_0\| + \frac{1}{n}\|y_0\| < \inf\{\|y\| : \ |x| \leq y,\ y \in E^+\}.$$

Put $z_0 = (1 + \frac{1}{n})y_0$. Consequently we have $z_0 \in A$, then

$$\inf\{\|y\| : \ |x| \leq y,\ y \in E^+\} < \|z_0\|,$$

which is impossible.

ii) First we show that

$$\inf\{\|y\| : \ |x| \leq y,\ y \in E\} = \sup\{\|z\| : \ z \leq |x|,\ z \in E\},$$

holds whenever $x \in G$. Set

$$A = \{z \leq |x| : \ z \in E^+\},$$

and

$$B = \{y \geq |x| : \ y \in E\}.$$

Since E is order dense and majorizing of G, it follows that A and B are not empty and they are directed sets. We consider the set A as a net $\{z_\alpha\}$, where $z_\alpha = \alpha$ for each $\alpha \in A$. In the same way we consider $B = \{y_\beta\}$, and by using [[2], Theorem 1.34], we write $z_\alpha \uparrow |x|$ and $y_\beta \downarrow |x|$. Since $z_\alpha \leq |x| \leq y_\beta$ for each α and β, it follows that $y_\beta - z_\alpha \downarrow 0$, and so

$$0 \leq \|y_\beta\| - \|z_\alpha\| \leq \|y_\beta - z_\alpha\| \to 0.$$

It follows that $\|x\|_t = \inf \|y_\beta\| = \sup \|z_\alpha\|$. Obviously that $\|.\|_t$ is a norm for G and $(G, \|.\|_t)$ is a normed lattice.

In Example 1, we note that c is neither Dedekind complete nor equipped with an order-continuous norm, yet we observe that $c^t = \ell^\infty$. However, Theorem 2 provides justification for extending the norm of E to a vector lattice E^t in various other cases.

It is also important to determine when $(E^t)^t = E^t$. In the following example, we demonstrate that E^t exists whenever E satisfies the Fatou property. It is worth noting that according to Example 4.3 and 4.4 from [1], every normed lattice with the Fatou property, in a general sense, is neither order-continuous nor Dedekind complete.

Example 2 By [[1], Theorem 4.12], if $(E, \|.\|)$ satisfies the Fatou property, the Dedekind completion of E, E^δ is a normed space with $\delta - norm$. Let E be the vector lattice of all real-valued functions defined on an infinite set X whose range is finite, with the pointwise ordering and satisfies the Fatou property. It can be seen that E is not Dedekind complete and $E^\delta = \ell^\infty(X)$.

We now present an important lemma that plays a crucial role throughout this manuscript.

Lemma 1 *Let E has order continuous norm. For each $0 \leq x \in E^t$, there are sequences $\{x_n\} \subseteq E^+$ and $\{y_n\} \subseteq E^+$ such that $x_n \uparrow x$, $x_n \xrightarrow{\|\cdot\|_t} x$, $y_n \downarrow x$ and $y_n \xrightarrow{\|\cdot\|_t} x$.*

Proof Choose $\{r_n\} \subseteq \mathbb{R}^+$ and $\{x_n\} \subseteq E^+$ satisfies in the following conditions:

1. $r_n \downarrow 0$,
2. $x_n \in \{z \in E: \ z \leq x \text{ and } \|x - z\|_t < r_n\}$, for each $n \in \mathbb{N}$,
3. $x_n \uparrow x$.

The justification for the above statement is as follows:
By [[2], Theorem 1.34], set

$$A = \{z \leq x: \ z \in E^+\} = \{z_\alpha\},$$

and

$$B = \{y \geq x: \ y \in E\} = \{y_\beta\},$$

such that $z_\alpha \uparrow x$ and $y_\beta \downarrow x$. Then $z_\alpha \leq x \leq y_\beta$ holds for each α and β. Thus

$$\|x - y_\beta\|_t, \|x - z_\alpha\|_t \leq \|z_\alpha - y_\beta\|_t = \|z_\alpha - y_\beta\| \to 0.$$

Let $0 < r_1 \in \mathbb{R}$. Then there exist

$$z_1 \in \{z \in A: \ \|x - z\|_t \leq r_1\},$$

and

$$0 < r_2 < \min\{r_1, \ \|z_1 - x\|_t\}.$$

We choose $z_2, z_3, \ldots, z_n$ and $z_{n+1} \in \{z \in A: \ \|x - z_n\|_t \leq r_n\}$ where

$$0 < r_n < \min\{r_{n-1}, \ \|z_{n-1} - x\|_t\}.$$

We define $x_n = \vee_{i=1}^n z_i$. Now, if $x_n \leqslant w \leqslant x$ for each $n \in \mathbb{N}$, then

$$0 \leqslant x - w \leqslant x - x_n \leq x - z_n.$$

It follows that

$$\|x - w\|_t \leqslant \|x - x_n\|_t \leqslant \|x - z_n\|_t \leqslant r_n \downarrow 0.$$

Thus $x = w$, and so $\sup x_n = x$. Therefore $x_n \uparrow x$ and $\|x_n - x\| \to 0$.
The existence of $\{y_n\}$ follows the same argument.

Theorem 3 *Suppose E is a normed lattice. If E is a KB-space or an AL-space, then E^t is also a KB-space or an AL-space, respectively.*

Proof Assume that $\{x_n\} \subseteq (E^t)^+$ is increasing sequence such that

$$\sup \|x_n\|_t < +\infty.$$

By using Lemma 1, for each $n \in \mathbb{N}$, there is increasing sequences

$$\{x_{n,m}\}_m \subseteq E^+,$$

such that $x_{n,m} \uparrow_m x_n$ and $\|x_n - x_{n,m}\|_t \xrightarrow{m} 0$. Take $y_n = \bigvee_{i,j=1}^n x_{i,j}$. It follows that $0 \leq y_n \uparrow$ and $\sup \|y_n\| \leq \sup_{i,j} \|x_{i,j}\| \leq \sup \|x_n\| < +\infty$. Since E is a KB-space, it follows that there exists $x \in E$ such that $\|y_n - x\|_t \to 0$. On the other hand, the inequalities $y_n \leq x_n \leq x$ implies that $\|x_n - x\|_t \leq \|y_n - x\|_t$ for each $n \in \mathbb{N}$. It follows that $\|x_n - x\|_t \to 0$ holds in E^t. Now, if E is an $AL-$space, then E has order continuous norm. Now, let $0 < x, y \in E^t$ with $x \wedge y = 0$. By using Lemma 1, there are $\{x_n\}$ and $\{y_n\}$ in E^+ such that $x_n \uparrow x$, $y_n \uparrow y$, $\|x - x_n\|_t \to 0$ and $\|y - y_n\|_t \to 0$. It follows that $0 \leqslant x_n \wedge y_n \uparrow x \wedge y = 0$ implies that $x_n \wedge y_n = 0$ for each $n \in \mathbb{N}$. Hence

$$\|x_n + y_n\| = \|x_n\| + \|y_n\|,$$

for each $n \in \mathbb{N}$. Then

$$\|x + y\|_t = \lim_n \|x_n + y_n\| = \lim_n \|x_n\| + \lim_n \|y_n\| = \|x_n\|_t + \|y\|_t.$$

Consequently, E^t is an $AL-$space.

Theorem 4 *For a normed lattice E with order continuous norm, we have the following assertions*

1. *If $\hat{E}$ is a norm completion of E, then $E^t \subseteq \hat{E} = E^u$, and if E is norm complete, then $E^t = E^u = E$.*
2. *For each $x \in E^t$ and $A \subseteq E$ with $\sup A = x$, we have $\|x\|_t = \sup_{z \in A} \|z\|$.*
3. *For each $x \in E^t$ and $A \subseteq E$ with $\inf A = x$, we have $\|x\|_t = \inf_{z \in A} \|z\|$.*
4. *$(E^t, \|.\|_t)$ has Fatou property and $B_{E^t} = \{x \in E^t : \ \|x\|_t \leq 1\}$ is order closed.*
5. *If E is an ideal in E^t, then $\hat{E} = E^t$.*

Proof 1. According to [[1], Theorem 2.40], $(\hat{E}, \widehat{\|.\|})$ is a normed lattice, where $\widehat{\|.\|}$ is the unique extension of the norm from E to $\hat{E}$. Let $x \in E^t$. Then by Lemma 1, there exists $\{x_n\}$ in E^+ such that $x_n \uparrow x^+$ and $\|x^+ - x_n\|_t \to 0$. Thus $\{x_n\}$ is a norm Cauchy sequence in E, and so convergence in $\hat{E}$. It follows that $x^+ \in \hat{E}$. In the similar way $x^- \in \hat{E}$, which implies that $x \in \hat{E}$. Now by Theorem 7.51 of [1], we conclude that $E^t \subseteq \hat{E} = E^u$ and $\|.\|_t = \widehat{\|.\|}$. On the other hand if E is norm complete, it is obvious that $E^t = E^u = E$ and $\|.\| = \|.\|_t = \widehat{\|.\|}$.

2. By [[1], Theorem 7.54], E^u has order continuous norm. Since by part (1), we have $E^t \subseteq E^u$, it follows that E^t has order continuous norm. Consider $A = (x_\alpha)$ with $\sup A = x$. It follows that $x - x_\alpha \downarrow 0$ which implies that $\|x - x_\alpha\|_t \to 0$. Then by using inequalities $0 \leq \|x\|_t - \|x_\alpha\| \leq \|x - x_\alpha\|_t$, we have $\sup_\alpha \|x_\alpha\| = \|x\|_t$.
3. The proof follows a similar argument as that of (2).
4. By [[1], Lemma 4.2], $(E, \|.\|)$ has Fatou property. The proof of the first statement follows a similar argument to that of Theorem 3(1), and we omit the details. The second part follows by [[1], Theorem 4.6].
5. The proof follows by [[1], Theorem 3.8].

Note that a linear subspace E of a partially ordered vector space G is said to be order dense if $x = \inf\{y \in E : \ x \leq y\}$ for every $x \in G$. Based on our earlier discussion, we can pose the following question:

Problem 1 If E^t is a partially ordered vector space and E is order dense and majorizing in E^t, is there a norm extension from $(E, \|\cdot\|)$ to E^t?

3 The extension of order bounded operators

In this section, we explore the extension properties of order-bounded operators. Specifically, we consider T to be an order-bounded operator from a normed lattice E into a Dedekind complete normed lattice F, and we aim to introduce an operator T^t from E^t to F as an extension of T. We investigate various lattice and topological properties of T^t that hold when these properties are satisfied by T. Our analysis provides insights into the behavior of order-bounded operators under extensions of normed lattices, which has important applications in the positive operators studying and related fields.

Theorem 5 *Let T be an order bounded operator from normed lattice E into Dedekind complete normed lattice F. We have the following assertions.*

1. *There exists an extension order bounded operator T^t from E^t into F satisfying $T^t(y) = Ty$ for each $y \in E$.*
2. *For each positive continuous operator T, we have $\|T\| = \|T^t\|$, and if T is norm continuous, then so is T^t.*
3. *$|T|^t = |T^t|$.*
4. *For each $T, S \in \mathcal{L}_b(E, F)$, we have $(T \vee S)^t = T^t \vee S^t$.*
5. *If $S : E^t \to F$ is an order bounded and norm continuous operator, then $T^t = S$.*
6. *Each order interval of E^t is $\sigma(E^t, (E^t)')$-compact.*

Proof 1. Since T is an order bounded operator and F Dedekind complete, we have $T = T^+ - T^-$. So first we assume that T is a positive operator from E into F. According to [[2], Theorem 1.32], the mapping $p : E^t \to F$ defined via the formula

$$p(x) = \inf\{Ty : \ y \in E, \ x \leqslant y\}, \ x \in E^t.$$

is a monotone sublinear and $Ty = p(y)$ for each $y \in E$. So by [[3], Theorem 1.5.7], there is an extension T^t from E^t into F satisfying $T^t x \leq p(x^+)$ for all $x \in E^t$, and $T^t y = Ty$ for all $y \in E$. Now we define $T^t = (T^+)^t - (T^-)^t$, and so for all $y \in E$, we have

$$T^t y = (T^+)^t(y) - (T^-)^t(y) = T^+ y - T^- y = Ty.$$

2. Assume that T is a positive operator and $x \in E^t$. According part (1), we have $T^t x \leq p(x^+) \leq Ty$ for all $y \in E$ such that $y \geq x^+$, and so $\|T^t x\| \leq \|Ty\|$ for all $y \in E$ such that $y \geq x^+$. It follows that

$$\|T^t x\| \leq \|T\| \inf_{y \geq x^+} \|y\| \leq \|T\| \|x^+\|_t \leq \|T\| \|x\|_t.$$

Then $\|T^t\| \leq \|T\|$. Since $B_E \subseteq B_{E^t}$, follows that $\|T\| \leqslant \|T^t\|$. Thus $\|T\| = \|T^t\|$, and proof follows.

3. In this part, we assume that x, y, z are members of E and x^t, y^t, z^t are members of E^t when there is not any confused. Now let $x^t \geq 0$. Since E is order dense in E^t, we have the following equalities

$$\begin{aligned}
(T^t)^+(x^t) &= \sup_{0 \leq y^t \leq x^t} T^t y^t \\
&= \sup_{0 \leq y^t \leq x^t} \sup_{0 \leq z \leq y^t} T^t z \\
&= \sup_{0 \leq y \leq x^t} Ty \\
&= \sup_{0 \leq z \leq x^t} \sup_{0 \leq y \leq z} Ty \\
&= \sup_{0 \leq z \leq x^t} T^+ z \\
&= (T^+)^t(x^t).
\end{aligned}$$

Similarly, we have $(T^t)^-(x^t) = (T^-)^t(x^t)$ for all $x^t \geq 0$. It is obvious that for each $x^t \in E^t$, we have $(T^t)^+ x^t = (T^+)^t(x^t)$ and $(T^t)^- x^t = (T^-)^t(x^t)$. Thus

$$|T|^t = (T^+ + T^-)^t = (T^+)^t + (T^-)^t = (T^t)^+ + (T^t)^- = |T^t|.$$

4. By using the equality $T \vee S = \frac{1}{2}(T + S + |T - S|)$ and part (3), proof follows.

5. First let $0 \leq x \in E^t$. By Lemma 1, there exists $\{x_n\}$ in E^+ such that $x_n \uparrow x^+$ and $\|x^+ - x_n\|_t \to 0$. Since $S^+ x_n \uparrow$ and $\|x^+ - x_n\| \to 0$, follows that $S^+ x_n \uparrow S^+ x$. We have $T = S|_E$ (restriction of S on E), which follows that $T^- = S^-|_E$ and $T^+ = S^+|_E$. Obviously $(T^-)^t = S^-$ and $(T^+)^t = S^+$, and so by part (3), we have the following equalities

$$S = S^+ - S^- = (T^+)^t - (T^-)^t = (T^t)^+ - (T^t)^- = T^t.$$

Thus $S = T^t$ on E^- and E^+, which follows that

$$Sx = Sx^+ - Sx^- = T^t x^+ - T^t x^- = T^t x,$$

for each $x \in E^t$.

6. Consider $a, b \in (E^t)^+$ and $a < b$. By Lemma 1, take $\{x_n\}$ and $\{y_n\}$ in E^+ such that $x_n \uparrow a$, $y_n \downarrow b$, $\|a - x_n\|_t \to 0$ and $\|y_n - b\|_t \to 0$. Since E has order continuous norm, $[x_n, y_n] \cap E$ is $\sigma(E, E')$-compact subset of E for each $n \in \mathbb{N}$. It follows that $[a, b] \cap E$ is $\sigma(E, E')$-compact subset of E. Now, if we set
$$V = \{s \in E : \; x'(s) < r \text{ and } x' \in E'\},$$
then by using part (5), the order density of V is
$$V^t = \{s \in E^t : \; (x')^t(s) < r \text{ and } (x')^t \in (E^t)'\}.$$
It is obvious that $V \subseteq V^t$, and so $\sigma(E, E') \subseteq \sigma(E^t, (E^t)')$. Since $[a, b] \cap E$ is order dense in $[a, b]$, follows that $[a, b]$ is $\sigma(E^t, (E^t)')$-compact subset of E^t.

In the following, we examine some properties of the operator T^t, and we demonstrate that T^t preserves certain lattice and topological properties when these properties hold for T.

Theorem 6 *Let $0 \leq T \in \mathcal{L}_n(E, F)$. Then we have the following assertions*

1. *If $0 \leq x \leq E^t$ and $\{x_\alpha\} \subseteq E^+$ with $x_\alpha \downarrow x$, then $Tx_\alpha \downarrow T^t x$.*
2. *If $T(x \wedge y) = Tx \wedge Ty$ for each $0 \leq x, y \in E$, then T^t is a lattice homomorphism from E^t into F and moreover $T^t \in \mathcal{L}_n(E^t, F)$.*
3. *If $0 \leq T : E \to E$ is a band-preserving operator, then $T^t : E^t \to E^t$ is also band-preserving.*
4. *If $T : E \to F$ is an order bounded operator that preserves disjointness, then $T^t : E^t \to F$ also preserves disjointness.*
5. *Suppose E has an order continuous norm. Then $\{Tx_n\}$ is norm convergent in F for every positive increasing norm-bounded sequence $\{x_n\}$ in E if and only if $\{T^t x_n\}$ is norm convergent in F for every positive increasing t-norm-bounded sequence $\{x_n\}$ in E^t.*

Proof 1. Let $\{x_\alpha\} \subseteq E^+$ such that $x_\alpha \downarrow x$. If $y \in E^+$ such that $x \leq y$, then $y \vee x_\alpha \downarrow y$ holds in E, and so by order continuity of $T : E \to F$ and Theorem 4 (3), we see that
$$Ty = \inf\{T(x_\alpha \vee y)\} \leq \inf Tx_\alpha \leq T^t x.$$
This easily implies that $Tx_\alpha \downarrow T^t x$.

2. Assume that $0 \leq x, y \in E^t$. We prove that $T^t(x \wedge y) = T^t x \wedge T^t y$. By [[2], Theorem 1.34], there are $\{x_\alpha\}$ and $\{y_\beta\}$ of E^+ such that $x_\alpha \downarrow x$ and $y_\beta \downarrow y$. It follows that $x_\alpha \wedge y_\beta \downarrow x \wedge y$. Then by order continuity of $T : E \to F$ and Theorem 4 (3), we have the following equalities,
$$\begin{aligned} T^t(x \wedge y) &= \inf\{T(x_\alpha \wedge y_\beta)\} = \inf\{T(x_\alpha) \wedge T(y_\beta)\} \\ &= \inf\{T(x_\alpha)\} \wedge \inf\{T(y_\beta)\} = T^t x \wedge T^t y. \end{aligned}$$

By combining Theorem 1.10 and Theorem 2.14 from [2] with Theorem 3, we can conclude that the mapping $T^t : (E^t)^+ \to (F^t)^+$ has a unique extension $T^t : (E^t) \to (F^t)$, which is a lattice homomorphism. Now, we will show that $T^t \in \mathcal{L}_n(E^t, F)$. Let $\{x_\alpha\} \subseteq (E^t)^+$ be such that $x_\alpha \downarrow 0$. Put

$$A = \{y \in E^+ :\ \exists \alpha \text{ such that } x_\alpha \leq y\}.$$

Since E majorizes E^t, it follows that A is not empty. By using Theorem 5 since T is positive, T^t is positive. Thus $\inf T(A) \geq \inf T^t x_\alpha \geq 0$ holds in F. Since $A \downarrow 0$ and $T \in \mathcal{L}_n(E, F)$, it follows that $\inf T(A) = 0$, and so $T^t x_\alpha \downarrow 0$.

3. Let $x, y \in E^t$ satisfying $|x| \wedge |y| = 0$. Assume that $(x_\alpha), (y_\beta) \subseteq E^+$ such that $x_\alpha \uparrow |x|$ and $y_\beta \uparrow |y|$. It follows that $(x_\alpha \wedge y_\beta) \uparrow |x| \wedge |y| = 0$, and so $x_\alpha \wedge y_\beta = 0$, by [[2], Theorem 2.36], follows that $|Tx_\alpha| \wedge y_\beta = 0$ for each α and β. Since $|Tx_\alpha| \wedge y_\beta \uparrow |Tx| \wedge |y|$, we have $|Tx| \perp |y|$, and so by another using [[2], Theorem 2.36], proof follows.
4. Let $x, y \in E^t$ satisfying $x \perp y$. Assume that $(x_\alpha), (y_\beta) \subseteq E^+$ such that $x_\alpha \uparrow |x|$ and $y_\beta \uparrow |y|$. It follows that $(x_\alpha \wedge y_\beta) \uparrow |x| \wedge |y| = 0$. Now since T preserve disjointness, follows that $Tx_\alpha \perp Tx_\beta$. From our hypothesis, we have $Tx_\alpha \wedge Tx_\beta \uparrow T^t|x| \wedge T^t|y|$ which follows that $T^t|x| \wedge T^t|y| = 0$. Since $|T^t x| \wedge |T^t y| \leq T^t|x| \wedge T^t|y|$, we have $T^t x \perp T^t y$.
5. Since $T = T^+ - T^-$, without loss generality, we assume that T is a positive operator. Assume that $\{x_n\} \subseteq (E^t)^+$ is increasing sequence with $\sup \|x_n\|_t < +\infty$. By using Lemma 1, for each $n \in \mathbb{N}$, there are positive increasing sequences $\{x_{n,m}\}_m$ with $x_{n,m} \uparrow_m x_n$ and $\|x_n - x_{n,m}\|_t \to 0$. Take $y_n = \bigvee_{i,j=1}^n x_{i,j}$. It follows that $0 \leq y_n \uparrow$ and

$$\sup \|y_n\| \leq \sup_{i,j} \|x_{i,j}\| \leq \sup \|x_n\| < +\infty.$$

By assumption there is $s^* \in F$ such that $\|Ty_n - s^*\| \to 0$. Then by using [[2], Theorem 2.46], $Ty_n \uparrow s^*$. By Theorem 5, we know that T^t is norm continuous from E^t into F. It follows that $\|T^t x_n - Tx_{n,m}\| \xrightarrow{m} 0$ holds in F. The inequality $Tx_{n,m} \leq Ty_n \leq T^t x_n$ implies that

$$\|T^t x_n - s^*\| \leq \|T^t x_n - Tx_{n,m}\| \text{ for each } n, m \in \mathbb{N}.$$

Then

$$\|T^t x_n - s^*\| \leq \|T^t x_n - Ty_n\| + \|Ty_n - s^*\| \to 0.$$

Thus $T^t x_n \to s^*$, and the proof follows.
The converse is straightforward.

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